Function Theory on a q-Analog of Complex Hyperbolic Space

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Abstract

This work deals with function theory on quantum complex hyperbolic spaces. The principal notions are expounded. We obtain explicit formulas for invariant integrals on 'finite' functions on a quantum hyperbolic space and on the associated quantum isotropic cone. Also we establish principal series of $U_q\mathfrak{su}_{n,m}$-modules related to this cone.

1 Introduction

Let us consider the group $SU_{n,m}$ of pseudo-unitary $(n+m)\times(n+m)$-matrices that preserve the following form in $\mathbb{C}^{n+m}$:

$$[x,y]=-x_1\bar{y}_1-\ldots-x_n\bar{y}_n+x_{n+1}\bar{y}_{n+1}+\ldots+x_{n+m}\bar{y}_{n+m}.$$

Then one can also consider the manifold $\widehat{\mathcal{H}}_{n,m}=\{x\in\mathbb{C}^{n+m}|[x,x]>0\}$ and its projectivization $\mathcal{H}_{n,m}$. The latter manifold is isomorphic to the homogeneous space $SU_{n,m}/S(U_{n,m-1}\times U_1)$, a complex hyperbolic space. There is a vast literature devoted to the study of these pseudo-Hermitian spaces of rank 1, in particular harmonic analysis on those (see J.Faraut [3], V.Molchanov [7, 8], G.van Dijk and Yu.Sharshov [2]).

In this paper we establish basic notions in the theory of quantum pseudo-Hermitian spaces. These objects initially appear in the work of Reshetikhin, Faddeev and Takhtadjan [9]. Later on the development of the theory of quantum bounded symmetric domains and quantum analogs of representation theory of noncompact real Lie groups made it clear that the above objects really worth studying. For example, the Penrose transform of the quantum matrix ball of rank 2 leads to a quantum analog of the complex hyperbolic space in $\mathbb{C}^4$, see [12].

We introduce a background of the function theory on quantum analogs of complex hyperbolic spaces $\mathcal{H}_{n,m}$ and of the related isotropic cones $\Xi_{n,m}=\{x\in\mathbb{C}^{n+m}|[x,x]=0\}$. We establish some special 'spaces of functions with compact support' (called finite functions, for short) and endow these noncommutative algebras with faithful representations. Then we introduce integrals on the spaces of finite functions and prove their invariance

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under the action of quantum universal enveloping algebra $U_q\mathfrak{su}_{n,m}$. Finally, we introduce a quantum analog of the principal (unitary) series of $U_q\mathfrak{su}_{n,m}$-modules related to a quantum analog of the cone Ξ.

These study were inspired and outlined by Leonid Vaksman some years ago. The authors are greatly indebted for him and D. Shklyarov for many helpful ideas towards this research.

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2 Preliminaries

Let $q \in (0,1)$. The Hopf algebra $U_q\mathfrak{sl}_N$ is given by its generators K_i, K_i^{-1}, E_i, F_i, $i = 1,2,\ldots,N-1$, and the relations:

$$\begin{gathered}
K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
K_iE_j = q^{a_{ij}}E_jK_i, \qquad K_iF_j = q^{-a_{ij}}F_jK_i,\\
E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}},\\
E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \qquad |i-j| = 1,\\
F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \qquad |i-j| = 1,\\
[E_i,E_j] = [F_i,F_j] = 0, \qquad |i-j| \neq 1.
\end{gathered}$$

The comultiplication Δ, the antipode S, and the counit ε are defined on the generators by

$$\begin{gathered}
\Delta(E_i) = E_i\otimes 1 + K_i\otimes E_i, \quad \Delta(F_i) = F_i\otimes K_i^{-1} + 1\otimes F_i, \quad \Delta(K_i) = K_i\otimes K_i,\\
S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1},\\
\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1,
\end{gathered}$$

see [4, Chapter 4].

We need also the Hopf algebra $\mathbb{C}[SL_N]_q$ of matrix elements of finite dimensional weight $U_q\mathfrak{sl}_N$-modules. Recall that $\mathbb{C}[SL_N]_q$ can be defined by the generators t_{ij}, $i,j = 1,...,N$, (the matrix elements of the vector representation in a weight basis) and the relations

$$\begin{aligned}
& t_{ij'}t_{ij''} = qt_{ij''}t_{ij'}, && j' < j'',\\
& t_{i'j}t_{i''j} = qt_{i''j}t_{i'j}, && i' < i'',\\
& t_{ij}t_{i'j'} = t_{i'j'}t_{ij}, && i < i' \;\&\; j > j',\\
& t_{ij}t_{i'j'} = t_{i'j'}t_{ij} + (q-q^{-1})t_{ij'}t_{i'j}, && i < i' \;\&\; j < j',
\end{aligned}$$

together with one more relation

$$\det{}_q\,\mathbf{t} = 1,$$

where $\det_q \mathbf{t}$ is a q-determinant of the matrix $\mathbf{t} = (t_{ij})_{i,j=1,\ldots,N}$:

$$\det{}_q\,\mathbf{t} = \sum_{s\in S_N}(-q)^{l(s)}t_{1s(1)}t_{2s(2)}\ldots t_{Ns(N)},$$

with $l(s) = \mathrm{card}\{(i,j)|i<j \ \&\ s(i)>s(j)\}$.

Let also $U_q\mathfrak{su}_{n,m}$, $m+n=N$, stand for the Hopf $*$-algebra $(U_q\mathfrak{sl}_N, *)$ given by

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \qquad E_j^* = \begin{cases} K_j F_j, & j \neq n, \\ -K_j F_j, & j = n, \end{cases} \qquad F_j^* = \begin{cases} E_j K_j^{-1}, & j \neq n, \\ -E_j K_j^{-1}, & j = n, \end{cases}$$

with $j = 1, \ldots, N-1$ [9, 11].

3 $*$-Algebra $\mathrm{Pol}\,(\mathcal{H}_{n,m})_q$

Let $m, n \in \mathbb{N}$, $m \geq 2$, and $N \stackrel{\mathrm{def}}{=} n+m$. Recall that the classical complex hyperbolic space $\mathcal{H}_{n,m}$ can be obtained by projectivization of the domain

$$\widehat{\mathcal{H}}_{n,m} = \left\{ (t_1, \ldots, t_N) \in \mathbb{C}^N \,\middle|\, -\sum_{j=1}^{n} |t_j|^2 + \sum_{j=n+1}^{N} |t_j|^2 > 0 \right\}.$$

Now we pass from the classical case $q=1$ to the quantum case $0<q<1$. Let us consider the well known [9] q-analog of the pseudo-Hermitian spaces. Let $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ stand for the unital $*$-algebra with the generators $t_1, t_2, \ldots, t_N$ and the commutation relations as follows:

$$\begin{aligned} & t_i t_j = q t_j t_i, \qquad i<j \\ & t_i t_j^* = q t_j^* t_i, \qquad i \neq j \\ & t_i t_i^* = t_i^* t_i + (q^{-2}-1) \sum_{k=i+1}^{N} t_k t_k^*, \qquad i>n \\ & t_i t_i^* = t_i^* t_i + (q^{-2}-1) \sum_{k=i+1}^{n} t_k t_k^* - (q^{-2}-1) \sum_{k=n+1}^{N} t_k t_k^*, \qquad i \leq n. \end{aligned} \tag{3.1}$$

It is important to note that

$$c = -\sum_{j=1}^{n} t_j t_j^* + \sum_{j=n+1}^{N} t_j t_j^*$$

is central in $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$. Moreover, c is not a zero divisor in $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$. This allows one to embed the $*$-algebra $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ into its localization $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c}$ with respect to the multiplicative system $c^{\mathbb{N}}$.

The $*$-algebra $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c}$ admits the following bigrading:

$$\deg t_j = (1,0), \qquad \deg t_j^* = (0,1), \qquad j = 1, 2 \ldots, N.$$

Introduce the notation

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c} \,\middle|\, \deg f = (0,0) \right\}.$$

This $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ will be called the algebra of regular functions on the quantum hyperbolic space.

We are going to endow the $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ with a structure of $U_q\mathfrak{su}_{n,m}$-module algebra [1]. For this purpose, we embed it into the $U_q\mathfrak{su}_{n,m}$-module $*$-algebra $\mathrm{Pol}\left(\widetilde{X}\right)_q$ of 'regular functions on the quantum principal homogeneous space' constructed in [11].

Recall that $\mathrm{Pol}\left(\widetilde{X}\right)_q \stackrel{\mathrm{def}}{=} (\mathbb{C}[SL_N]_q, *)$, with $\mathbb{C}[SL_N]_q$ being the well-known algebra of regular functions on the quantum group SL_N, and the involution $*$ being defined by

$$t_{ij}^* = \mathrm{sign}[(i-m-1/2)(n-j+1/2)](-q)^{j-i}\det\nolimits_q T_{ij}.$$

Here $\det_q$ stands for the quantum determinant [1], and the matrix T_{ij} is derived from the matrix $T=(t_{kl})$ by discarding its i's row and j's column.

It follows from $\det_q T = 1$ that

$$-\sum_{j=1}^{n} t_{1j}t_{1j}^* + \sum_{j=n+1}^{N} t_{1j}t_{1j}^* = 1.$$

Thus the map $J : t_j \mapsto t_{1j}$, $j=1,2,\ldots,N$, admits a unique extension to a homomorphism of $*$-algebras $J : \mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c} \to \mathrm{Pol}\left(\widetilde{X}\right)_q$. Its image will be denoted by $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$. It is easy to verify that the $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is *embedded* this way into $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ and its image is just the subalgebra in $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ generated by $t_{1j}t_{1k}^*$, $j,k=1,2,\ldots,N$. In what follows we will identify $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ with its image under the map J.

R e m a r k 3.1 1. $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ can be characterized in two ways. Firstly,

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widetilde{X}\right)_q \,\middle|\, \triangle_L(f) = 1 \otimes f \right\}.$$

Here $\triangle_L$ is the coaction $\triangle_L : \mathrm{Pol}\left(\widetilde{X}\right)_q \to \mathbb{C}[\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})]_q \otimes \mathrm{Pol}\left(\widetilde{X}\right)_q$, $\triangle_L : t_{ij} \mapsto \sum\limits_{k=1}^{N} \pi(t_{ik}) \otimes t_{kj}$, and $\pi : \mathrm{Pol}\left(\widetilde{X}\right)_q \to \mathbb{C}[\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})]_q$ is the factorization map with respect to the two-sided ideal in $\mathrm{Pol}\left(\widetilde{X}\right)_q$ generated by t_{1k}, t_{k1}, $k=2,3,\ldots,N$, cf. [5, 11.6.2, 11.6.4].

2. Another characterization is in observing that $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is the subalgebra of $U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})$-invariants under the left action in $\mathrm{Pol}\left(\widetilde{X}\right)_q$. The latter action is a dual to the coaction $\triangle_L$ as in [5, 1.3.5, Proposition 15]. To prove the equivalence one should observe the $U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})$-invariance of $t_{1j}t_{1k}^*$ and compare the dimensions of graded components of the algebras $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ and $\mathbb{C}[GL_N]_q^{U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})}$.

We use the notation t_j instead of t_{1j} for the generators of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$.

Let I_φ, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, be the $*$-automorphism of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ defined on the generators $\{t_j\}_{j=1,\dots,N}$ by

$$I_\varphi : t_j \mapsto e^{i\varphi} t_j. \tag{3.2}$$

Then one more description of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is as follows:

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q \middle| \; I_\varphi(f) = f \quad \text{for all} \quad \varphi \right\}.$$

At the end of this section we are going to produce explicit formulas for the action of $U_q\mathfrak{su}_{n,m}$ on $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$.

The action of $U_q\mathfrak{su}_{n,m}$ on $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ is described as follows:

$$E_j t_i = \begin{cases} q^{-1/2} t_{i-1}, & j+1 = i, \\ 0, & \text{otherwise}, \end{cases}$$
$$F_j t_i = \begin{cases} q^{1/2} t_{i+1}, & j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$
$$K_j^{\pm 1} t_i = \begin{cases} q^{\pm 1} t_i, & j = i, \\ q^{\mp 1} t_i, & j+1 = i, \\ t_i, & \text{otherwise}, \end{cases}$$

$$E_j t_i^* = \begin{cases} -q^{-3/2} t_{i+1}^*, & j = i \;\&\; i \neq n, \\ q^{-3/2} t_{i+1}^*, & j = i \;\&\; i = n, \\ 0, & \text{otherwise}, \end{cases}$$
$$F_j t_i^* = \begin{cases} -q^{3/2} t_{i-1}^*, & j+1 = i \;\&\; i \neq n+1, \\ q^{3/2} t_{i-1}^*, & j+1 = i \;\&\; i = n+1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$
$$K_j^{\pm 1} t_i^* = \begin{cases} q^{\mp 1} t_i^*, & j = i, \\ q^{\pm 1} t_i^*, & j+1 = i, \\ t_i, & \text{otherwise}. \end{cases}$$

4 A $*$-Algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ of finite functions

Let us produce a faithful $*$-representation T of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ in a pre-Hilbert space $\mathcal{H}$ (the method of constructing T is well known; see, for example, [11]).

The space $\mathcal{H}$ is a linear span of its orthonormal basis $\{e(i_1, i_2, \dots, i_{N-1}) |\, i_1, \dots, i_n \in -\mathbb{Z}_+;\ i_{n+1}, \dots, i_{N-1} \in \mathbb{N}\}$.

The $*$-representation T is a restriction to $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ of the $*$-representation of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ defined by

$$\begin{aligned} T(t_j)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(q^{2(i_j-1)}-1\right)^{1/2} e(i_1,\ldots,i_j-1,\ldots,i_{N-1}),\\ T(t_j^*)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(q^{2i_j}-1\right)^{1/2} e(i_1,\ldots,i_j+1,\ldots,i_{N-1}), \end{aligned} \tag{4.1}$$

for $j \le n$,

$$\begin{aligned} T(t_j)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(1-q^{2(i_j-1)}\right)^{1/2} e(i_1,\ldots,i_j-1,\ldots,i_{N-1}),\\ T(t_j^*)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(1-q^{2i_j}\right)^{1/2} e(i_1,\ldots,i_j+1,\ldots,i_{N-1}), \end{aligned} \tag{4.2}$$

for $n < j < N$, and, finally,

$$\begin{aligned} T(t_N)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1,\ldots,i_{N-1}),\\ T(t_N^*)e(i_1,\ldots,i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1,\ldots,i_{N-1}). \end{aligned} \tag{4.3}$$

Define the elements $\{x_j\}_{j=1,\ldots,N}$ as follows:

$$x_j \stackrel{\mathrm{def}}{=} \begin{cases} \sum\limits_{k=j}^{N} t_k t_k^*, & j > n,\\ -\sum\limits_{k=j}^{n} t_k t_k^* + \sum\limits_{k=n+1}^{N} t_k t_k^*, & j \le n. \end{cases} \tag{4.4}$$

Obviously, $x_1 = 1$, $x_i x_j = x_j x_i$,

$$t_j x_k = \begin{cases} q^2 x_k t_j, & j < k,\\ x_k t_j, & j \ge k, \end{cases} \tag{4.5}$$

hence

$$t_j^* x_k = \begin{cases} q^{-2} x_k t_j^*, & j < k,\\ x_k t_j^*, & j \ge k. \end{cases} \tag{4.6}$$

The vectors $e(i_1,\ldots,i_{N-1})$ are joint eigenvectors of the operators $T(x_j)$, $j = 1,2,\ldots,N$:

$$\begin{aligned} &T(x_1) = I,\\ &T(x_j)e(i_1,\ldots,i_{N-1}) = q^{2\sum\limits_{k=1}^{j-1} i_k} e(i_1,\ldots,i_{N-1}). \end{aligned} \tag{4.7}$$

The joint spectrum of the pairwise commuting operators $T(x_j)$, $j = 1,2,\ldots,N$, is

$$\begin{aligned} \mathfrak{M} = \bigl\{ &(x_1,\ldots,x_N) \in \mathbb{R}^N \bigm| \\ &x_i/x_j \in q^{2\mathbb{Z}} \ \&\ 1 = x_1 \le x_2 \le \ldots \le x_{n+1} > x_{n+2} > \ldots > x_N > 0 \bigr\}. \end{aligned}$$

Proposition 4.1 *T is a faithful representation of* $\mathrm{Pol}(\mathcal{H}_{n,m})_q$.

Proof. It suffices to verify faithfulness of the (unrestricted) representation T of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$. It is quite obvious that an arbitrary element of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ can be written as a finite sum

$$f=\sum_{(i_1,\ldots,i_N,j_1,\ldots,j_N):\,i_kj_k=0} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}(x_2,\ldots,x_N)t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1},$$

where $f_{IJ}(x_2,\ldots,x_N)$ are polynomials, $I=(i_1,\ldots,i_N)$, $J=(j_1,\ldots,j_N)$. It follows from the definition of T that every summand

$$t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}(x_2,\ldots,x_N)t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}$$

takes a basis vector $e(k_1,\ldots,k_{N-1})$ to a scalar multiple of the basis vector $e(k_1+j_1-i_1,\ldots,k_n+j_n-i_n,k_{n+1}-j_{n+1}+i_{n+1},\ldots,k_{N-1}-j_{N-1}+i_{N-1})$. Moreover, the sets of indices $(k_1+j_1-i_1,\ldots,k_{N-1}-j_{N-1}+i_{N-1})$ of the image basis vectors are different for different monomials, provided the indices of the initial monomial $e(k_1,\ldots,k_{N-1})$ have modules large enough. Therefore, to prove our claim, it suffices to choose arbitrarily a summand of f and to find an initial basis vector $e(k_1,\ldots,k_{N-1})$ in such a way that the chosen summand does not annihilate (under T) the vector $e(k_1,\ldots,k_{N-1})$.

Let us consider a basis vector $e(k_1,\ldots,k_{N-1})$ with $|k_s|>j_s$ for all $s=1,\ldots,N-1$. Then

$$\begin{aligned}T\left(t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}\right)&e(k_1,\ldots,k_{N-1})=\\&\mathrm{const}\cdot e(k_1+j_1,\ldots,k_n+j_n,k_{n+1}-j_{n+1},\ldots,k_{N-1}-j_{N-1}),\end{aligned}$$

where $\mathrm{const}\neq 0$.

Moreover, $T(f_{IJ}(x_2,\ldots,x_N))$ acts by multiplying the basis vector by a (value of a) polynomial $p\left(q^{2k_1},\ldots,q^{2k_{N-1}}\right)$ (due to (4.7)), where $p(u_1,u_2,\ldots,u_{N-1})=f_{IJ}(u_1,u_1u_2,\ldots,u_1u_2\cdots u_{N-1})$, and p is certainly a nonzero polynomial. A routine argument allows one to find $k_1,\ldots,k_{N-1}$ such that $|k_s|>j_s$ and $p\left(q^{2k_1},\ldots,q^{2k_{N-1}}\right)\neq 0$. This proves the claim we need. $\square$

Let P be the orthogonal projection of $\mathcal{H}$ onto the linear span of vectors $\{e(\underbrace{0,\ldots,0}_{n},i_{n+1},\ldots,i_{N-1})|\,i_{n+1},\ldots,i_{N-1}\in\mathbb{N}\}$. Of course $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ does not contain an element f_0 such that $T(f_0)=P$. Our immediate intention is to add f_0 with this property.

Consider the $*$-algebra $\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right)\supset\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ derived from $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ by adding an element f_0 to its list of generators and the relations as below to its list of relations:

$$\begin{aligned}&t_j^*f_0=f_0t_j=0,\qquad j\le n,\\&x_{n+1}f_0=f_0x_{n+1}=f_0,\\&f_0^2=f_0^*=f_0,\\&t_jf_0=f_0t_j;\qquad t_j^*f_0=f_0t_j^*,\qquad j\ge n+1.\end{aligned}\tag{4.8}$$

The relation $I_\varphi f_0=f_0$ allows one to extend the $*$-automorphism I_φ (3.2) of the algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ to the $*$-automorphism of $\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right)$. Let

$$\mathrm{Fun}(\mathcal{H}_{n,m})\stackrel{\mathrm{def}}{=}\left\{f\in\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right)\middle|\ I_\varphi f=f\right\}.$$

Obviously, there exists a unique extension of the $*$-representation T to a $*$-representation of the $*$-algebra $\mathrm{Fun}(\mathcal{H}_{n,m})$ such that $T(f_0)=P$.

Our subsequent observations involve extensively the two-sided ideal $\mathcal{D}(\mathcal{H}_{n,m})_q$ of $\mathrm{Fun}(\mathcal{H}_{n,m})$ generated by f_0. We call this ideal the algebra of finite functions on the quantum hyperbolic space.

Theorem 4.2 *The representation T of $\mathcal{D}(\mathcal{H}_{n,m})_q$ is faithful.*

Proof. Obviously, every $f\in\mathcal{D}(\mathcal{H}_{n,m})_q$ admits a unique decomposition

$$f=\sum_{\substack{(i_1\ldots,i_N,j_1\ldots j_N):\\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N=\\ =j_1+\ldots+j_n+i_{n+1}+\ldots+i_N}} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_0t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}.$$

A straightforward application of the commutation relations (4.8) allows us to refine the above decomposition as follows:

$$f=\sum_{\substack{(i_1\ldots,i_N,j_1\ldots j_N):\ i_kj_k=0\ \&\\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N=\\ =j_1+\ldots+j_n+i_{n+1}+\ldots+i_N}} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}, \tag{4.9}$$

where

$$f_{IJ}=\sum_K p_K(x_{n+2},\ldots,x_{N-1})t_1^{k_1}t_2^{k_2}\ldots t_n^{k_n}f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1} \tag{4.10}$$

for some nonzero polynomials p_K.

Let us consider a basis vector $e(a_1,\ldots,a_{N-1})$. Every summand from (4.9) takes $e(a_1,\ldots,a_{N-1})$ to a scalar multiple of the vector $e(a_1+j_1-i_1,\ldots,a_n+j_n-i_n,a_{n+1}-j_{n+1}+i_{n+1},\ldots,a_{N-1}-j_{N-1}+i_{N-1})$ (nonzero if well defined). By our assumptions on entries of I and J, the subset of nonzero multiples as above are linearly independent. Thus it suffices to choose arbitrarily a summand in (4.9) and to prove that it does not annihilate some basis vector.

Let us also choose arbitrarily a summand

$$p_K(x_{n+2},\ldots,x_{N-1})t_1^{k_1}t_2^{k_2}\ldots t_n^{k_n}f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1}$$

from (4.10). Now $T(f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1})T(t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1})e(a_1,\ldots,a_{N-1})=\mathrm{const}\cdot e(0,\ldots,0,a_{n+1}-j_{n+1},\ldots,a_{N-1}-j_{N-1})$. Here const $=0$ unless $a_s+k_s+j_s=0$ for $s=1,\ldots,n$ and $a_s>j_s$ for $s=n+1,\ldots,N-1$. Set $a_s=-k_s-j_s$ for $s=1,\ldots,n$.

Now let us consider the action of $T(p_K(x_{n+2},\ldots,x_{N-1}))$ on vectors of the form $e(-k_1,\ldots,-k_n,a_{n+1}-j_{n+1},\ldots,a_{N-1}-j_{N-1})$ with $a_s>j_s$ for $s=n+1,\ldots,N-1$. An argument similar to that used in the final paragraph of the proof of Proposition 4.1 allows us to choose $a_{n+1},\ldots,a_{N-1}$ in such a way that $T\left(t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}\right)$ does not annihilate $e(a_1,\ldots,a_{N-1})$. This proves our claim. □

R e m a r k 4.3 i) Due to (4.8), f_0 can be treated as a function of x_{n+1}:

$$f_0 = f_0(x_{n+1}) = \begin{cases} 1, & x_{n+1} = 1, \\ 0, & x_{n+1} \in q^{-2\mathbb{N}}. \end{cases} \tag{4.11}$$

(Recall that $\operatorname{spec} x_{n+1} = q^{-2\mathbb{Z}_+}$). Thus f_0 is a q-analog of the characteristic function of the submanifold

$$\left\{ (t_1, \ldots, t_N) \in \mathbb{C}^N \middle|\; t_1 = t_2 = \ldots = t_n = 0 \right\} \cap \mathcal{H}_{n,m}.$$

ii) Let $f(x_{n+1})$ be a polynomial. Then it follows from (4.4), (4.5) that

$$\sum_{i=1}^{n} t_i f(x_{n+1}) t_i^* = f(q^2 x_{n+1}) \sum_{i=1}^{n} t_i t_i^* = f\left(q^2 x_{n+1}\right)(x_{n+1} - 1). \tag{4.12}$$

This computation, together with (4.11), allows one to consider the element $f_1 = \sum\limits_{i=1}^{n} t_i f_0 t_i^*$ as a function of x_{n+1} such that

$$f_1(x_{n+1}) = \begin{cases} q^{-2} - 1, & x_{n+1} = q^{-2}, \\ 0, & x_{n+1} = 1 \ \text{ or } \ x_{n+1} \in q^{-2\mathbb{N}-2}. \end{cases}$$

Thus a multiple application of (4.12) leads to the following claim: $\mathcal{D}(\mathcal{H}_{n,m})_q$ contains *all* finite functions of x_{n+1} (i.e., such functions f that $f(q^{-n}) = 0$ for all but finitely many $n \in \mathbb{N}$).

Our intention now is to endow $\mathcal{D}(\mathcal{H}_{n,m})_q$ with a structure of $U_q\mathfrak{su}_{n,m}$-module algebra. For that, it suffices to describe the action of the operators $\{E_j, F_j, K_j\}_{j=1,\ldots,N-1}$ on f_0. Here it is:

$$E_n f_0 = -\frac{q^{-1/2}}{q^{-2} - 1} t_n f_0 t_{n+1}^*, \tag{4.13}$$

$$F_n f_0 = -\frac{q^{3/2}}{q^{-2} - 1} t_{n+1} f_0 t_n^*, \tag{4.14}$$

$$K_n f_0 = f_0, \tag{4.15}$$

$$E_j f_0 = F_j f_0 = (K_j - 1) f_0 = 0, \qquad j \neq n. \tag{4.16}$$

R e m a r k 4.4 To see that the above structure of $U_q\mathfrak{su}_{n,m}$-module algebra on $\mathcal{D}(\mathcal{H}_{n,m})_q$ is well-defined, it suffices to use an argument contained in [11]. Here we restrict ourselves to explaining the motives which lead to (4.13) – (4.16). An application of (3.3), (3.4) and (4.4) allows one to conclude that for any polynomial $f(t)$

$$E_n f(x_{n+1}) = q^{-1/2} t_n \frac{f(q^{-2} x_{n+1}) - f(x_{n+1})}{q^{-2} x_{n+1} - x_{n+1}} t_{n+1}^*, \tag{4.17}$$

$$F_n f(x_{n+1}) = q^{3/2} t_{n+1} \frac{f(q^{-2} x_{n+1}) - f(x_{n+1})}{q^{-2} x_{n+1} - x_{n+1}} t_n^*, \tag{4.18}$$

$$E_j f = F_j f = (K_j - 1) f = 0 \ \text{ for } \ j \neq n,\ j = 1, 2, \ldots, N-1. \tag{4.19}$$

A subsequent application of (4.17) – (4.19) to the non-polynomial function f_0 (4.11) yields (4.13) – (4.16).

5 Invariant integral

The aim of this section is to present an explicit formula for a positive invariant integral on the space of finite functions $\mathcal{D}(\mathcal{H}_{n,m})_q$ and thereby to establish its existence.

Let $\nu_q : \mathcal{D}(\mathcal{H}_{n,m})_q \to \mathbb{C}$ be a linear functional defined by

$$\nu_q(f) = \mathrm{Tr}(T(f) \cdot Q) = \int\limits_{\mathcal{H}_{n,m}} f d\nu_q, \tag{5.1}$$

where $Q : \mathcal{H} \to \mathcal{H}$ stands for the linear operator given on the basis elements $e(i_1, \ldots, i_{N-1})$ by

$$Qe(i_1, \ldots, i_{N-1}) = \mathrm{const} \cdot q^{2\sum\limits_{j=1}^{N-1}(N-j)i_j} e(i_1, \ldots, i_{N-1}), \qquad \mathrm{const} > 0. \tag{5.2}$$

Thus $Q = \mathrm{const} \cdot T(x_2 \cdot \ldots \cdot x_N)$; this follows from (4.7).

Theorem 5.1 *The functional ν_q determined by* (5.1) *is well defined, positive, and $U_q\mathfrak{su}_{n,m}$-invariant.*

Proof. It follows from (3.1), (4.4), (4.5) that any element f of the algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ can be written in a unique way in the form

$$f = \sum_{\substack{(i_1 \ldots, i_N, j_1 \ldots j_N) :\ i_k j_k = 0\ \& \\ i_1 + \ldots + i_n + j_{n+1} + \ldots + j_N = \\ = j_1 + \ldots + j_n + i_{n+1} + \ldots + i_N}} t_1^{i_1} \ldots t_n^{i_n} t_{n+1}^{*i_{n+1}} \ldots t_N^{*i_N} f_{IJ}(x_2, \ldots, x_N) t_N^{j_N} \ldots t_{n+1}^{j_{n+1}} t_n^{*j_n} \ldots t_1^{*j_1}, \tag{5.3}$$

with $f_{IJ}(x_2, \ldots, x_N)$ being a polynomial in $x_2, \ldots, x_n, x_{n+2}, \ldots, x_N$ and a *finite* function in x_{n+1}, that is, $f_{IJ}(x_2, \ldots, x_N)$ has the form

$$\sum_{\text{finite sum}} \alpha_{\mathbb{K}} x_2^{k_2} \cdots x_n^{k_n} f_{\mathbb{K}}(x_{n+1}) x_{n+2}^{k_{n+2}} \cdots x_N^{k_N}, \qquad \alpha_{\mathbb{K}} \in \mathbb{C}, \tag{5.4}$$

and $f_{\mathbb{K}}(q^{-2l}) \neq 0$ for finitely many $l \in \mathbb{Z}_+$.

Then, by our definition,

$$\nu_q : f \mapsto \mathrm{const} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots+2i_{N-1}}\right) \cdot$$

$$\cdot\, q^{2(N-1)i_1+2(N-2)i_2+\ldots+2i_{N-1}}, \tag{5.5}$$

and for f of the form (5.4) the series in the right hand side of (5.5) converges.

The positivity of the linear functional ν_q means that

$$\nu_q(f^*f) > 0 \qquad \text{for} \quad f \neq 0.$$

This follows from the explicit formula (5.5) and the *faithfulness* of the $*$-representation T of the algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ (see Section 4).

What remains is to establish the $U_q\mathfrak{su}_{n,m}$-invariance of ν_q. The desired invariance is equivalent to

$$\nu_q(E_j f) = 0, \qquad \nu_q(F_j f) = 0. \tag{5.6}$$

for any $f \in \mathcal{D}(\mathcal{H}_{n,m})_q$ and $j = 1, 2, \ldots, N-1$. Observe that ν_q is a real functional, i.e., $\nu_q(f^*) = \overline{\nu_q(f)}$. The latter relation follows from selfadjointness of the operator $Q : \mathcal{H} \to \mathcal{H}$ involved in the definition of ν_q. This allows us to reduce the proof of (5.6) to proving the abridged version of it

$$\nu_q(E_j f) = 0, \qquad j = 1, 2, \ldots, N-1. \tag{5.7}$$

We are going to establish (5.7) for $j \le n$; for other j the proof is similar.

Moreover, for a function f of the form

$$f = t_1^{i_1} \ldots t_n^{i_n} t_{n+1}^{*i_{n+1}} \ldots t_N^{*i_N} f_{IJ}(x_2, \ldots, x_N) t_N^{j_N} \ldots t_{n+1}^{j_{n+1}} t_n^{*j_n} \ldots t_1^{*j_1}$$

with $i_k j_k = 0$ for $k = 1, 2, \ldots, N$, one has $\nu_q(E_j f) = 0$ provided $I \ne (0, \ldots, \underset{(j+1)\text{th place}}{0, 1, 0}, \ldots, 0)$ and $J \ne (0, \ldots, \underset{j\text{th place}}{0, 1, 0}, \ldots, 0)$ (if $j < n$) or $I \ne (0, 0, \ldots, 0)$ and $J \ne (0, \ldots, \underset{j\text{th}\,(j+1)\text{th places}}{0, 1, 1, 0}, \ldots, 0)$ (if $j = n$). Thus we have to verify that $\nu_q\left(E_j\left(t_{j+1} f(x_2, \ldots, x_N) t_j^*\right)\right) = 0$.

It can be demonstrated by a direct computation that for $j \le n$

$$\begin{aligned} & E_j(t_{j+1} f(x_2, \ldots, x_N) t_j^*) = \\ & \qquad = q^{-1/2} \left[q^2 f(x_2, \ldots, x_j, q^2 x_{j+1}, \ldots, q^2 x_N)(x_{j+1} - x_j) \frac{q^{-2} x_{j+2} - x_{j+1}}{(1-q^2) x_{j+1}} \right. \\ & \qquad\qquad \left. - f(x_2, \ldots, x_{j+1}, q^2 x_{j+2}, \ldots, q^2 x_N)(x_{j+2} - x_{j+1}) \frac{q^{-2} x_{j+1} - x_j}{(1-q^2) x_{j+1}} \right]. \end{aligned} \tag{5.8}$$

1. Let $j = n$. Then

$$\begin{aligned} & \nu_q\left(E_j\left(t_{j+1} f(x_2, \ldots, x_N) t_j^*\right)\right) = \\ & \quad = \text{const'} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} \left[f\left(q^{2i_1}, \ldots, q^{2i_1+\ldots+2i_{n-1}}, q^{2i_1+\ldots+2i_n+2}, \ldots, q^{2i_1+\ldots+2i_{N-1}+2}\right) \cdot \right. \\ & \qquad \cdot \frac{q^2\left(q^{2i_1+\ldots+2i_n} - q^{2i_1+\ldots+2i_{n-1}}\right)\left(q^{2i_1+\ldots+2i_{n+1}-2} - q^{2i_1+\ldots+2i_n}\right)}{q^{2i_1+\ldots+2i_n}} - \\ & \qquad - f\left(q^{2i_1}, \ldots, q^{2i_1+\ldots+2i_n}, q^{2i_1+\ldots+2i_{n+1}+2}, \ldots, q^{2i_1+\ldots+2i_{N-1}+2}\right) \cdot \\ & \qquad \left. \cdot \frac{\left(q^{2i_1+\ldots+2i_{n+1}} - q^{2i_1+\ldots+2i_n}\right)\left(q^{2i_1+\ldots+2i_n-2} - q^{2i_1+\ldots+2i_{n-1}}\right)}{q^{2i_1+\ldots+2i_n}} \right] q^{2(N-1)i_1+\ldots+2i_{N-1}} = \end{aligned}$$

$$\begin{aligned} & = \text{const'} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} \left[f\left(q^{2i_1}, \ldots, q^{2i_1+\ldots+2i_{n-1}}, q^{2i_1+\ldots+2i_n+2}, \ldots, q^{2i_1+\ldots+2i_{N-1}+2}\right) \cdot \right. \\ & \qquad \cdot q^2\left(q^{2i_n} - 1\right)\left(q^{2i_{n+1}-2} - 1\right) - \\ & \qquad - f\left(q^{2i_1}, \ldots, q^{2i_1+\ldots+2i_n}, q^{2i_1+\ldots+2i_{n+1}+2}, \ldots, q^{2i_1+\ldots+2i_{N-1}+2}\right) \cdot \\ & \qquad \left. \cdot \left(q^{2i_{n+1}} - 1\right)\left(q^{2i_n-2} - 1\right) \right] q^{2i_1+\ldots+2i_{n-1}} q^{2(N-1)i_1+\ldots+2i_{N-1}}. \end{aligned}$$

Let us consider the inner sum (in i_n and i_{n+1}). For brevity, we denote $f\left(q^{2i_1},\ldots,q^{2i_1+\ldots+2i_{n-1}},q^{2i_1+\ldots+2i_n+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2}\right)$ by $\psi_{i_n+1,i_{n+1}}$.

$$\sum_{\substack{i\in-\mathbb{Z}_+\\ j\in\mathbb{N}}}\left[\psi_{i+1,j}\cdot q^2\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)-\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)\right]q^{2(N-n)i+2(N-n-1)j}=$$

$$=\sum_{i\in-\mathbb{Z}_+,\,j\in\mathbb{N}}\psi_{i+1,j}\cdot\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j+2}$$
$$-\sum_{i\in-\mathbb{Z}_+,\,j\in\mathbb{N}}\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$=q^{-2(N-n-1)}\sum_{i\le 1,j\in\mathbb{N}}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$-q^{-2(N-n-1)}\sum_{i\in-\mathbb{Z}_+,j\ge 2}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n)j}=0.$$

Thus the proof in this case is complete.

2. Let $j<n$.

$$\sum_{i,j\in-\mathbb{Z}_+}\left[\psi_{i+1,j}\cdot q^2\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)-\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)\right]q^{2(N-n)i+2(N-n-1)j}=$$
$$=q^{-2(N-n-1)}\sum_{i\le 1,j\in-\mathbb{Z}_+}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$-q^{-2(N-n-1)}\sum_{i\in-\mathbb{Z}_+,j\le 1}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n)j}=0.$$

The Theorem is proved. □

R e m a r k 5.2 It is reasonable to choose const in (5.2) so that the following normalization property is valid:

$$\nu_q(f_0)=1.$$

This allows us to find the constant explicitly:

$$\mathrm{const}=q^{-(2N-n-2)(N-n-1)}\prod_{j=n+1}^{N-1}\left(1-q^{2(N-j)}\right).$$

6 Quantum homogeneous space $\Xi_{n,m}$

Let $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ stand for the quotient algebra of $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q$ by the ideal $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q\cdot c$ (recall that c belongs to the center of $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q$). This is a q-analog of the polynomial algebra on the isotropic cone. Define an automorphism I_φ, $\varphi\in\mathbb{R}/2\pi\mathbb{Z}$, of the algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ by

$$I_\varphi(t_j)=e^{i\varphi}t_j,\qquad I_\varphi(t_j^*)=e^{-i\varphi}t_j^*.$$

Then it follows from the definition that

$$\mathrm{Pol}(\Xi_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q \middle| \; I_\varphi(f) = f \quad \text{for any} \quad \varphi \right\}.$$

We are going to produce a $*$-representation T_0 of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ in a pre-Hilbert space $\mathcal{H}_0$ in such a way that the restriction of T_0 to the subalgebra $\mathrm{Pol}(\Xi_{n,m})_q$ is a faithful $*$-representation of $\mathrm{Pol}(\Xi_{n,m})_q$.

Let $\{e(i_1, i_2, \ldots, i_{N-1}) |\, i_1 \in \mathbb{Z}; i_2, \ldots, i_n \in -\mathbb{Z}_+;\; i_{n+1}, \ldots, i_{N-1} \in \mathbb{N}\}$ be the orthonormal basis of the space $\mathcal{H}_0$. Then T_0 is defined as follows.

$$\begin{aligned} T_0(t_1)e(i_1, \ldots, i_{N-1}) &= q^{i_1-1} e(i_1 - 1, \ldots, i_{N-1}), \\ T_0(t_1^*)e(i_1, \ldots, i_{N-1}) &= q^{i_1} e(i_1 + 1, \ldots, i_{N-1}), \end{aligned} \tag{6.1}$$

$$\begin{cases} T_0(t_j)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(q^{2(i_j-1)} - 1\right)^{1/2} e(i_1, \ldots, i_j - 1, \ldots, i_{N-1}), \\ T_0(t_j^*)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(q^{2i_j} - 1\right)^{1/2} e(i_1, \ldots, i_j + 1, \ldots, i_{N-1}), \\ \qquad \text{for} \quad 1 < j \le n, \end{cases} \tag{6.2}$$

$$\begin{cases} T_0(t_j)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(1 - q^{2(i_j-1)}\right)^{1/2} e(i_1, \ldots, i_j - 1, \ldots, i_{N-1}), \\ T_0(t_j^*)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(1 - q^{2i_j}\right)^{1/2} e(i_1, \ldots, i_j + 1, \ldots, i_{N-1}), \\ \qquad \text{for} \quad n < j < N, \end{cases} \tag{6.3}$$

$$\begin{aligned} T_0(t_N)e(i_1, \ldots, i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1, \ldots, i_{N-1}), \\ T_0(t_N^*)e(i_1, \ldots, i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1, \ldots, i_{N-1}), \end{aligned} \tag{6.4}$$

Let us introduce the notation

$$\xi_j = \begin{cases} \displaystyle\sum_{k=j}^{N} t_k t_k^*, \quad j > n, \\ \displaystyle -\sum_{k=j}^{n} t_k t_k^* + \sum_{k=n+1}^{N} t_k t_k^*, \quad j \le n. \end{cases}$$

Evidently, $\xi_1 = 0$, and the elements $\xi_2, \ldots, \xi_N$ satisfy (4.5) – (4.6) with x_k being replaced by ξ_k. The joint spectrum of the pairwise commuting operators $\{T_0(\xi_j)\}_{j=\overline{1,N}}$ is the set

$$\mathfrak{M}_0 = \big\{(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \big| \\ \xi_j \in q^{2\mathbb{Z}},\; j > 1 \;\&\; 0 = \xi_1 \le \xi_2 \le \ldots \le \xi_{n+1} > \xi_{n+2} > \ldots > \xi_N > 0 \big\}.$$

Similarly to the case of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$, any element from $\mathrm{Pol}(\Xi_{n,m})_q$ can be written in the form

$$f = \sum_{\substack{IJ=0 \\ \text{finite sum} \\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N= \\ =i_{n+1}+\ldots+i_N+j_1+\ldots+j_n}} t_1^{i_1}\ldots t_n^{i_n} t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N} f_{IJ}(\xi_2,\ldots,\xi_N) t_N^{j_N}\ldots t_{n+1}^{j_{n+1}} t_n^{*j_n}\ldots t_1^{*j_1},$$

where f_{IJ} are polynomials in $\xi_2,\ldots,\xi_N$, and such decomposition is unique.

The $*$-algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ is a $U_q\mathfrak{su}_{n,m}$-module algebra. Namely, the action of $U_q\mathfrak{su}_{n,m}$ on the generators t_j, t_j^* of $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ is defined by (3.3) – (3.4). This definition is correct due to the fact that the element c of the covariant algebra $\mathrm{Pol}(\widetilde{\mathcal{H}}_{n,m})_q$ is $U_q\mathfrak{su}_{n,m}$-invariant. Thus the $*$-algebra $\mathrm{Pol}(\Xi_{n,m})_q$ is a $U_q\mathfrak{su}_{n,m}$-module algebra too. The same computations as in the case of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ show that for any polynomial $f(t)$

$$\begin{aligned} E_n f(\xi_{n+1}) &= q^{-1/2} t_n \frac{f(q^{-2}\xi_{n+1}) - f(\xi_{n+1})}{q^{-2}\xi_{n+1} - \xi_{n+1}} t_{n+1}^*, \\ F_n f(\xi_{n+1}) &= q^{3/2} t_{n+1} \frac{f(q^{-2}\xi_{n+1}) - f(\xi_{n+1})}{q^{-2}\xi_{n+1} - \xi_{n+1}} t_n^*, \\ (K_n-1)f(\xi_{n+1}) &= E_j f(\xi_{n+1}) = F_j f(\xi_{n+1}) = (K_j-1)f(\xi_{n+1}) = 0, \quad j \neq n. \end{aligned} \tag{6.5}$$

Now (4.5), (4.6), and (6.5) allow one to introduce the covariant $*$-algebra $\mathcal{D}(\Xi_{n,m})$ of finite functions on the quantum homogeneous space $\Xi_{n,m}$. It is formed by elements of the form (5.3) with ξ_k instead of x_k, where $f_{IJ}(\xi_2,\ldots,\xi_N)$ are polynomials of $\xi_2,\ldots,\xi_n,\xi_{n+2},\ldots,\xi_N$ and finite functions of ξ_{n+1} (i.e., f_{IJ} has the form (5.4) where $f_{\mathbb{K}}(q^{2l}) \neq 0$ for finitely many $l \in \mathbb{Z}$).

Theorem 6.1 *T_0 can be extended to a faithful $*$-representation of the $*$-algebra $\mathcal{D}(\Xi_{n,m})$.*

R e m a r k 6.2 The algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ has the same list of generators as $\mathrm{Pol}(\widetilde{\Xi})_q$ while the lists of relations differ by replacing $c-1=0$ with $c=0$. Furthermore, the differences between the formulas (4.1) – (4.3) and (6.1) – (6.4) are low enough to enable us to apply the same argument in proving Theorems 6.1 and 4.2.

Our intention now is to produce an invariant integral on $\mathcal{D}(\Xi_{n,m})$. Denote by ν_q^0 the linear functional $\nu_q^0 : \mathcal{D}(\Xi_{n,m}) \to \mathbb{C}$ given by

$$\nu_q^0(f) = \mathrm{Tr}(T_0(f)\cdot Q_0) \quad \left(= \int_{\Xi_{n,m}} f d\nu_q^0 \right) \tag{6.6}$$

with $Q_0 : \mathcal{H}_0 \to \mathcal{H}_0$ being the linear map given by

$$Q_0 e(i_1,\ldots,i_{N-1}) = \mathrm{const}\cdot q^{2\sum_{j=1}^{N-1}(N-j)i_j} e(i_1,\ldots,i_{N-1}). \tag{6.7}$$

Theorem 6.3 *The functional ν_q^0 is well-defined, positive, and $U_q\mathfrak{su}_{n,m}$-invariant.*

Proof. It follows from the definition that

$$\nu_q^0(f) = \mathrm{const} \cdot \sum_{\substack{i_1 \in \mathbb{Z} \\ (i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots+2i_{N-1}}\right) q^{2i_1(N-1)+\ldots+2i_{N-1}}. \tag{6.8}$$

Here f_{00} is the function involved in the decomposition (5.3) of f.

To prove that the definition (6.6) of ν_q^0 is correct, it now suffices to show that the series in the r.h.s. of (6.8) is absolutely convergent for f_{00} satisfying the condition

$$f_{00}\left(\xi_2, \ldots, \xi_n, q^{2l}, \xi_{n+2}, \ldots, \xi_N\right) = 0 \quad \text{for } l \neq l_0.$$

Let f_{00} be such a function. Then

$$\sum_{\substack{i_1 \in \mathbb{Z} \\ (i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots+2i_n} q^{2i_1+\ldots+2i_{N-1}}\right) q^{2i_1(N-1)+\ldots+2i_{N-1}} =$$

$$= \sum_{\substack{(i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2l_0-2i_2-\ldots-2i_n}, q^{2l_0-2i_3-\ldots-2i_n}, \ldots, q^{2l_0-2i_n}, q^{2l_0}, q^{2l_0+2i_{n+1}}, \ldots\right) \cdot$$

$$\cdot\, q^{2l_0(N-1)} \cdot q^{2i_1(N-1)+\ldots+2i_{N-1}} \cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n} \cdot q^{2i_{n+1}(m-1)+i_{n+2}(m-2)+\ldots+2i_{N-1}}. \tag{6.9}$$

It is implicit here that only terms with $i_1+\ldots+i_n = l_0$ can be non-zero; also, the following obvious equality is used:

$$q^{2(N-1)i_1+\ldots+2i_{N-1}} = q^{2i_1} \cdot q^{2i_1+2i_2} \cdot \ldots \cdot q^{2i_1+\ldots+2i_{N-1}}.$$

Now to establish the convergence of the series (6.9), it suffices to recall that f_{00} is a polynomial in $\xi_2, \ldots, \xi_n, \xi_{n+2}, \ldots, \xi_N$.

The positive definiteness of ν_q^0 can be explained in the same way as it was done in section 5 for ν_q.

Let us turn to proving the invariance of ν_q^0. To do this, one needs to reproduce the proof of a similar fact for ν_q almost literally, including the computations of cases 1 and 2. But now there is one more case to be considered:

3. Let $j = 1$, then (see (5.8))

$$E_1(t_2 f(\xi_2, \ldots, \xi_N) t_1^*) =$$
$$= q^{-1/2}\left[f(q^2\xi_2, \ldots, q^2\xi_N)\frac{\xi_2(\xi_3 - q^2\xi_2)}{(1-q^2)\xi_2} - f(\xi_2, q^2\xi_3, \ldots, q^2\xi_N)\frac{q^{-2}\xi_2(\xi_3-\xi_2)}{(1-q^2)\xi_2}\right] =$$
$$= \frac{q^{-1/2}}{1-q^2}\left[f(q^2\xi_2, \ldots, q^2\xi_N)(\xi_3 - q^2\xi_2) - q^{-2} f(\xi_2, q^2\xi_3, \ldots, q^2\xi_N)(\xi_3 - \xi_2)\right].$$

Now let us show that $\nu_q^0(E_1(t_2 f(\xi_2,\ldots,\xi_N)t_1^*))=0$. In fact,

$$\nu_q^0(E_1(t_2 f(\xi_2,\ldots,\xi_N)t_1^*)) =$$
$$= \mathrm{const}' \cdot \sum_{\substack{i_1\in\mathbb{Z}\\ (i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1}\\ (i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}} \Big[f(q^{2i_1+2},q^{2i_1+2i_2+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})(q^{2i_2-2}-1)q^{2i_1+2}$$
$$-f(q^{2i_1},q^{2i_1+2i_1+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})q^{-2}(q^{2i_2}-1)q^{2i_1}\Big]\, q^{2i_1(N-1)+\ldots+2i_{N-1}}. \quad (6.10)$$

As usual, we denote $f(q^{2i_1+2},q^{2i_1+2i_2+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})$ by ψ_{i_1+1,i_2}. Let us compute the inner sum over i_1 and i_2 in the r.h.s. of (6.10).

$$\sum_{i\in\mathbb{Z},j\in-\mathbb{Z}_+}\left[q^2\psi_{i+1,j}(q^{2j-2}-1)-q^{-2}\psi_{i,j+1}(q^{2j}-1)\right]\cdot q^{2iN}q^{2j(N-2)} =$$
$$= \sum_{i\in\mathbb{Z},j\in-\mathbb{Z}_+}\psi_{i,j}(q^{2j-2}-1)\cdot q^{2iN+2jN-4j-2N+2} - \sum_{i\in\mathbb{Z},j\leq 1}\psi_{i,j}(q^{2j-2}-1)\cdot q^{2iN+2j(N-2)-2N+2}=0. \ \square$$

R e m a r k 6.4 Here const is chosen in (6.7) so that the following normalization property is valid:

$$\nu_q^0(f_0)=1.$$

This allows us to find the constant explicitly:

$$\mathrm{const}=q^{-(N-n)(N-n-1)}\prod_{j=1}^{n-1}\left(1-q^{2j}\right)\prod_{j=1}^{N-n-1}\left(1-q^{2j}\right).$$

7 Principal non-unitary and unitary series of representations of $U_q\mathfrak{su}_{n,m}$ related to the space $\Xi_{n,m}$

The element ξ_{n+1} quasi-commutes with all the generators of the algebra $\mathrm{Pol}(\Xi_{n,m})_q$. Thus $(\xi_{n+1})^{\mathbb{Z}_+}$ is an Ore set and one can consider a localization $\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}$ of the algebra $\mathrm{Pol}(\Xi_{n,m})_q$ with respect to the multiplicative system $(\xi_{n+1})^{\mathbb{Z}_+}$. Evidently, the $U_q\mathfrak{su}_{n,m}$-module algebra structure extends to the localization in a unique way.

Denote by γ the automorphism of the algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ given on the generators by

$$\gamma: t_j\mapsto qt_j, \qquad t_j^*\mapsto qt_j^*.$$

Note that γ is well defined due to the homogeneity of the defining relations for $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$. Obviously, $\gamma(\xi_{n+1})=q^2\xi_{n+1}$, and this allows one to extend γ to an automorphism of the algebra $\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}$, which commutes with the action of $U_q\mathfrak{su}_{n,m}$. This can be deduced from (3.3), (3.4), and (6.5).

Set

$$\mathcal{E}(\Xi_{n,m})_q=\{f\in\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}|\gamma(f)=f\}.$$

Thus $\mathcal{E}(\Xi_{n,m})_q$ is a $U_q\mathfrak{su}_{n,m}$-submodule in $\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}$.

Now we introduce representations of principal series related to the quantum cone. Let $s \in \mathbb{Z}$. The representation π_s is defined as follows:

$$\pi_s(\eta)f = \eta(f \cdot \xi_{n+1}^{s-N+1})\xi_{n+1}^{-(s-N+1)}, \qquad f \in \mathcal{E}(\Xi_{n,m})_q,\ \eta \in U_q\mathfrak{su}_{n,m}.$$

Now we can consider the operator-valued functions $\pi_s(\cdot)$ as Laurent polynomials in the variable $u = q^s$. These polynomials are uniquely determined at integer values of s. Thus there exist unique 'analytic continuation' of such polynomials, so we obtain $U_q\mathfrak{su}_{n,m}$-modules of principal series related to the quantum cone for arbitrary $s \in \mathbb{C}$. In the following we will denote by $\mathcal{E}_s(\Xi_{n,m})_q$ the space $\mathcal{E}(\Xi_{n,m})_q$ endowed with the π_s-action of $U_q\mathfrak{su}_{n,m}$.

Our immediate intention is to produce an invariant integral in $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$.

Note that $\mathcal{D}(\Xi_{n,m})_q$ can be made a covariant $\mathcal{E}(\Xi_{n,m})_q$-bimodule using the relations (4.5), (4.6).

Let $\chi_l \in \mathcal{D}(\Xi_{n,m})_q$ stand for the function of ξ_{n+1} such that

$$\chi_l(q^{2k}) = \delta_{kl}, \qquad k, l \in \mathbb{Z}.$$

Lemma 7.1 *For any $f \in \mathcal{E}_{-N+1}(\Xi_{n,m})_q$, the integral*

$$b_q^{(l)}(f) \stackrel{\mathrm{def}}{=} \int\limits_{\Xi_{n,m}} f \cdot \chi_l d\nu_q^0 \tag{7.1}$$

does not depend on l.

Proof.

$$\begin{aligned}
&b_q^{(l)}(f) = \\
&= \mathrm{const} \sum_{\substack{i_1 \in \mathbb{Z} \\ (i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}(q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots+2i_{N-1}})\chi_l(q^{2i_1+\ldots+2i_{N-1}})q^{2i_1(N-1)+\ldots+2i_{N-1}} = \\
&= \mathrm{const} \sum_{\substack{(i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}(q^{2l-2i_2-\ldots-2i_n}, q^{2l-2i_3-\ldots-2i_n}, \ldots, q^{2l-2i_n}, q^{2l}, q^{2l+2i_{n+1}}, \ldots)\cdot \\
&\qquad \cdot q^{2l(N-1)} \cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}}.
\end{aligned} \tag{7.2}$$

Clearly, $f \in \mathcal{E}_{-N+1}(\Xi_{n,m})_q$ implies

$$\gamma(f_{00}(\xi_2, \ldots, \xi_N)) = q^{-2N+2} f_{00}(\xi_2, \ldots, \xi_N),$$

or, equivalently,

$$f_{00}(q^2\xi_2, \ldots, q^2\xi_N) = q^{-2N+2} f_{00}(\xi_2, \ldots, \xi_N),$$

and thus the r.h.s. of (7.2) can be rewritten as follows

$$\mathrm{const} \sum_{\substack{(i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} q^{2l(N-1)} f_{00}(q^{-2i_2-\ldots-2i_n}, q^{-2i_3-\ldots-2i_n}, \ldots, q^{-2i_n}, 1, q^{2i_{n+1}}, \ldots) \cdot$$

$$\cdot\, q^{2l(N-1)} \cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}} =$$

$$= \mathrm{const} \sum_{\substack{(i_2 \ldots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}(q^{-2i_2-\ldots-2i_n}, q^{-2i_3-\ldots-2i_n}, \ldots, q^{-2i_n}, 1, q^{2i_{n+1}}, \ldots) \cdot$$

$$\cdot\, q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}}. \ \Box \qquad (7.3)$$

Introduce the notation $b_q(f)$ or $\int f db_q$ for the linear functional (7.1) on $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. It follows from the proof of Lemma 7.1 that

$$b_q(f) = (q^{-2}-1)^N \cdot$$

$$\cdot \sum_{\substack{(j_1 \ldots, j_{n-1}) \in (-\mathbb{Z}_+)^{n-1} \\ (i_1, \ldots, i_{m-1}) \in \mathbb{N}^{m-1}}} f_{00}(q^{2j_1+\ldots+2j_{n-1}}, q^{2i_2+\ldots+2j_{n-1}}, \ldots, q^{-2j_{n-1}}, 1, q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots,2i_{m-1}}) \cdot$$

$$\cdot\, q^{2j_1+4j_2+\ldots+2(n-1)j_{n-1}} \cdot q^{2(m-1)i_1+2(m-2)i_2+\ldots+2i_{m-1}}. \qquad (7.4)$$

Theorem 7.2 *b_q is an invariant integral on $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$.*

Proof. By (6.5), the functions of ξ_{n+1} are $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_m)$-invariants. Thus b_q is a $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_m)$-invariant functional (see Theorem 6.3). It remains to prove that $b_q(F_n f) = b_q(E_n f) = 0$ for $f \in \mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Let us prove just one of these two equalities, for example, $b_q(E_n f) = \int\limits_{\Xi_{n,m}} E_n f \cdot \chi_l d\nu_q^0 = 0$.

The invariance of ν_q^0 and the fact that $\mathcal{D}(\Xi_{n,m})_q$ is a covariant $\mathcal{E}(\Xi_{n,m})_q$-bimodule imply that

$$b_q(E_n f) = -q^{-1} \int f \cdot E_n \chi_l d\nu_q^0, \qquad f \in \mathcal{E}_{-N+1}(\Xi_{n,m})_q$$

(the integration by parts is used here, see [1, Chapter 4]).

By (6.5),

$$
\begin{aligned}
-q^{-1}\int f\cdot E_n\chi_l d\nu_q^0 &= -q^{-1}\int f\cdot q^{-1/2}t_n\frac{\chi_l(q^{-2}\xi_{n+1})-\chi_l(\xi_{n+1})}{(q^{-2}-1)\xi_{n+1}}t_{n+1}^*d\nu_q^0=\\
&=-\frac{q^{-3/2}}{(q^{-2}-1)}\int f\cdot t_n\frac{\chi_{l+1}(\xi_{n+1})-\chi_l(\xi_{n+1})}{\xi_{n+1}}t_{n+1}^*d\nu_q^0=\\
&=-\frac{q^{-3/2}}{(q^{-2}-1)}\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}t_{n+1}^*\right)Q_0\right]=\\
&=-\frac{q^{-3/2}}{(q^{-2}-1)}(q^{-2}-1)^N\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}t_{n+1}^*\xi_2\xi_3\ldots\xi_N\right)\right]=\\
&=\operatorname{const}(q,n,N)\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}\xi_2\xi_3\ldots\xi_N t_{n+1}^*\right)\right]=\\
&=\operatorname{const}(q,n,N)\operatorname{Tr}\left[T_0\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)\xi_2\xi_3\ldots\xi_N\right)\right]=\\
&=\operatorname{const}'(q,n,N)\operatorname{Tr}\left[T_0\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)Q_0\right)\right]=\\
&=\operatorname{const}'(q,n,N)\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)d\nu_1^0. \qquad (7.5)
\end{aligned}
$$

If $f\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$, one has $t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Thus the latter expression in (7.5) can be rewritten as follows:

$$
\begin{aligned}
\operatorname{const}'(q,n,N)&\left(\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\chi_{l+1}d\nu_1^0-\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\chi_l d\nu_1^0\right)=\\
&=\operatorname{const}'(q,n,N)\left(b_q^{(l+1)}\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\right)-b_q^l\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\right)\right).
\end{aligned}
$$

It follows from Lemma 7.1 that the latter difference is zero. □

If $f_1\in\mathcal{E}_s(\Xi_{n,m})_q$ and $f_2\in\mathcal{E}_{-s}(\Xi_{n,m})_q$, one has $f_1\cdot f_2\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Now an application of the standard arguments (see, e.g., [1, Chapter 4]) which set correspondence between invariant integrals and invariant pairings, yields

Corollary 7.3 *The pairing* $\mathcal{E}_s(\Xi_{n,m})_q\times\mathcal{E}_{-s}(\Xi_{n,m})_q\to\mathbb{C}$,

$$
(f_1,f_2)\mapsto\langle f_1,f_2\rangle\stackrel{\text{def}}{=}\int f_1f_2db_q
$$

is $U_q\mathfrak{su}_{n,m}$*-invariant.*

Obviously, the involution $*$ of the $*$-algebra $\mathcal{E}(\Xi_{n,m})_q$ maps $\mathcal{E}_{i\lambda}(\Xi_{n,m})_q$ to $\mathcal{E}_{-i\lambda}(\Xi_{n,m})_q$ for $\lambda\in\mathbb{R}$.

Proposition 7.4 *The sesquilinear form*

$$
(f_1,f_2)=\int f_2^*f_1db_q,\qquad f_1,f_2\in\mathcal{E}_{i\lambda}(\Xi_{n,m})_q, \qquad (7.6)
$$

is invariant and positive definite.

Proof. The invariance follows immediately from Corollary 7.3 (the standard arguments from [1, Chapter 4] are to be applied here again).

To see that the form (7.6) is positive definite, one should recall that the integral ν_q^0 is positive definite (Theorem 6.3), and use the following computations:

$$(f,f)=\int f^*f db_q=\int\limits_{\Xi_{n,m}} f^*f\chi_l d\nu_q^0=\mathrm{Tr}\left(T_0\left(f^*f\chi_l\right)Q_0\right)=\mathrm{Tr}\left(T_0\left(f^*f\chi_l\chi_l\right)Q_0\right)=$$

$$=\mathrm{Tr}\left(T_0\left(f^*f\chi_l\cdot\mathrm{const}\cdot\xi_2\ldots\xi_N\chi_l\right)\right)=\mathrm{Tr}\left(T_0\left(\chi_l f^*f\chi_l\right)Q_0\right)=\mathrm{Tr}\left(T_0\left(\chi_l^*f^*f\chi_l\right)Q_0\right)=$$

$$=\int\limits_{\Xi_{n,m}}(f\chi_l)^*f\chi_l d\nu_q^0.$$

Here $f\in\mathcal{E}_{i\lambda}(\Xi_{n,m})_q$, $\lambda\in\mathbb{Z}$, and the obvious relations $\chi_l^2=\chi_l$, $\chi_l^*=\chi_l$, $\chi_l\xi_k=\xi_k\chi_l$ are used.

Thus $\mathcal{E}_{i\lambda}(\Xi_{n,m})_q$, $\lambda\in\mathbb{R}$, are unitary $U_q\mathfrak{su}_{n,m}$-modules. They will be called the modules of the principal unitary series related to $\Xi_{n,m}$.

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Function Theory on a q-Analog of Complex Hyperbolic Space

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Abstract

This work deals with function theory on quantum complex hyperbolic spaces. The principal notions are expounded. We obtain explicit formulas for invariant integrals on 'finite' functions on a quantum hyperbolic space and on the associated quantum isotropic cone. Also we establish principal series of $U_q\mathfrak{su}_{n,m}$-modules related to this cone, and obtain the necessary conditions for those modules to be equivalent.

1 Introduction

Let us consider the group $SU_{n,m}$ of pseudo-unitary $(n+m)\times(n+m)$-matrices that preserve the following form in $\mathbb{C}^{n+m}$:

$$[x,y] = -x_1\bar{y}_1 - \ldots - x_n\bar{y}_n + x_{n+1}\bar{y}_{n+1} + \ldots + x_{n+m}\bar{y}_{n+m}.$$

Then one can also consider the manifold $\widehat{\mathcal{H}}_{n,m} = \{x \in \mathbb{C}^{n+m} | [x,x] > 0\}$ and its projectivization $\mathcal{H}_{n,m}$. The latter manifold is isomorphic to the homogeneous space $SU_{n,m}/S(U_{n,m-1}\times U_1)$, a complex hyperbolic space. There is a vast literature devoted to the study of these pseudo-Hermitian spaces of rank 1, in particular harmonic analysis on those (see J.Faraut [4], V.Molchanov [7, 8], G.van Dijk and Yu.Sharshov [2]).

In this paper we establish basic notions in the theory of quantum pseudo-Hermitian spaces. These objects initially appear in the work of Reshetikhin, Faddeev and Takhtadjan [9]. Later on the development of the theory of quantum bounded symmetric domains and quantum analogs of representation theory of noncompact real Lie groups made it clear that the above objects really worth studying. For example, the Penrose transform of the quantum matrix ball of rank 2 leads to a quantum analog of the complex hyperbolic space in $\mathbb{C}^4$, see [12].

We introduce a background of the function theory on quantum analogs of complex hyperbolic spaces $\mathcal{H}_{n,m}$ and of the related isotropic cones $\Xi_{n,m} = \{x \in \mathbb{C}^{n+m} | [x,x] = 0\}$. We establish some special 'spaces of functions with compact support' (called finite functions, for short) and endow these noncommutative algebras with faithful representations. Then we introduce integrals on the spaces of finite functions and prove their invariance under the action of quantum universal enveloping algebra $U_q\mathfrak{su}_{n,m}$. Finally, we introduce

a quantum analog of the principal (unitary) series of $U_q\mathfrak{su}_{n,m}$-modules related to a quantum analog of the cone Ξ. For these modules we establish the necessary conditions for the equivalence.

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2 Preliminaries

Let $q \in (0,1)$. The Hopf algebra $U_q\mathfrak{sl}_N$ is given by its generators K_i, K_i^{-1}, E_i, F_i, $i = 1, 2, \ldots, N-1$, and the relations:

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_i = q^2E_iK_i, \qquad K_iF_i = q^{-2}F_iK_i,$$

$$K_iE_j = q^{-1}E_jK_i, \qquad K_iF_j = qF_jK_i, \qquad |i-j| = 1,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \qquad |i-j| = 1,$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \qquad |i-j| = 1,$$

$$[E_i, E_j] = [F_i, F_j] = 0, \qquad |i-j| \neq 1.$$

The comultiplication Δ, the antipode S, and the counit ε are defined on the generators by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1,$$

see [5, Chapter 4].

We need also the Hopf algebra $\mathbb{C}[SL_N]_q$ of matrix elements of finite dimensional weight $U_q\mathfrak{sl}_N$-modules. Recall that $\mathbb{C}[SL_N]_q$ can be defined by the generators t_{ij}, $i, j = 1, ..., N$, (the matrix elements of the vector representation in a weight basis) and the relations

$$\begin{aligned}
&t_{ij'}t_{ij''} = qt_{ij''}t_{ij'}, && j' < j'',\\
&t_{i'j}t_{i''j} = qt_{i''j}t_{i'j}, && i' < i'',\\
&t_{ij}t_{i'j'} = t_{i'j'}t_{ij}, && i < i' \;\&\; j > j',\\
&t_{ij}t_{i'j'} = t_{i'j'}t_{ij} + (q - q^{-1})t_{ij'}t_{i'j}, && i < i' \;\&\; j < j',
\end{aligned}$$

together with one more relation

$$\det\nolimits_q \mathbf{t} = 1,$$

where $\det_q \mathbf{t}$ is a q-determinant of the matrix $\mathbf{t} = (t_{ij})_{i,j=1,\ldots,N}$:

$$\det\nolimits_q \mathbf{t} = \sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} t_{2s(2)} \ldots t_{Ns(N)},$$

with $l(s) = \mathrm{card}\{(i,j) | i < j \ \& \ s(i) > s(j)\}$.

Let also $U_q\mathfrak{su}_{n,m}$, $m+n=N$, denotes the Hopf $*$-algebra $(U_q\mathfrak{sl}_N, *)$ given by

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \qquad E_j^* = \begin{cases} K_j F_j, & j \neq n, \\ -K_j F_j, & j = n, \end{cases} \qquad F_j^* = \begin{cases} E_j K_j^{-1}, & j \neq n, \\ -E_j K_j^{-1}, & j = n, \end{cases}$$

with $j = 1, \ldots, N-1$ [9, 11].

3 $*$-Algebra $\mathrm{Pol}\,(\mathcal{H}_{n,m})_q$

Let $m, n \in \mathbb{N}$, $m \geq 2$, and $N \stackrel{\mathrm{def}}{=} n+m$. Recall that the classical complex hyperbolic space $\mathcal{H}_{n,m}$ can be obtained by projectivization of the domain

$$\widehat{\mathcal{H}}_{n,m} = \left\{ (t_1, \ldots, t_N) \in \mathbb{C}^N \,\middle|\, -\sum_{j=1}^{n} |t_j|^2 + \sum_{j=n+1}^{N} |t_j|^2 > 0 \right\}.$$

Now we pass from the classical case $q=1$ to the quantum case $0 < q < 1$. Let us consider the well known [9] q-analog of the pseudo-Hermitian spaces. Let $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ denotes the unital $*$-algebra with the generators $t_1, t_2, \ldots, t_N$ and the commutation relations as follows:

$$\begin{aligned} & t_i t_j = q t_j t_i, \qquad i < j \\ & t_i t_j^* = q t_j^* t_i, \qquad i \neq j \\ & t_i t_i^* = t_i^* t_i + (q^{-2} - 1) \sum_{k=i+1}^{N} t_k t_k^*, \qquad i > n \\ & t_i t_i^* = t_i^* t_i + (q^{-2} - 1) \sum_{k=i+1}^{n} t_k t_k^* - (q^{-2} - 1) \sum_{k=n+1}^{N} t_k t_k^*, \qquad i \leq n. \end{aligned} \tag{3.1}$$

It is important to note that

$$c = -\sum_{j=1}^{n} t_j t_j^* + \sum_{j=n+1}^{N} t_j t_j^*$$

is central in $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$. Moreover, c is not a zero divisor in $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$. This allows one to embed the $*$-algebra $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ into its localization $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c}$ with respect to the multiplicative system $c^{\mathbb{N}}$.

The $*$-algebra $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c}$ admits the following bigrading:

$$\deg t_j = (1,0), \qquad \deg t_j^* = (0,1), \qquad j = 1, 2 \ldots, N.$$

Introduce the notation

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c} \,\middle|\, \deg f = (0,0) \right\}.$$

This $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ will be called the algebra of regular functions on the quantum hyperbolic space.

We are going to endow the $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ with a structure of $U_q\mathfrak{su}_{n,m}$-module algebra [1]. For this purpose, we embed it into the $U_q\mathfrak{su}_{n,m}$-module $*$-algebra $\mathrm{Pol}\left(\widetilde{X}\right)_q$ of 'regular functions on the quantum principal homogeneous space' constructed in [11].

Recall that $\mathrm{Pol}\left(\widetilde{X}\right)_q \stackrel{\mathrm{def}}{=} (\mathbb{C}[SL_N]_q, *)$, with $\mathbb{C}[SL_N]_q$ being the well-known algebra of regular functions on the quantum group SL_N, and the involution $*$ being defined by

$$t_{ij}^* = \mathrm{sign}[(i-m-1/2)(n-j+1/2)](-q)^{j-i}\det_q T_{ij}.$$

Here $\det_q$ is the quantum determinant [1], and the matrix T_{ij} is derived from the matrix $T=(t_{kl})$ by discarding its i's row and j's column.

It follows from $\det_q T = 1$ that

$$-\sum_{j=1}^{n} t_{1j}t_{1j}^* + \sum_{j=n+1}^{N} t_{1j}t_{1j}^* = 1.$$

Thus the map $J: t_j \mapsto t_{1j}$, $j=1,2,\ldots,N$, admits a unique extension to a homomorphism of $*$-algebras $J: \mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_{q,c} \to \mathrm{Pol}\left(\widetilde{X}\right)_q$. Its image will be denoted by $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$. It is easy to verify that the $*$-algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is *embedded* this way into $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ and its image is just the subalgebra in $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ generated by $t_{1j}t_{1k}^*$, $j,k=1,2,\ldots,N$. In what follows we will identify $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ with its image under the map J.

R e m a r k 3.1 1. $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ can be characterized in two ways. Firstly,

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widetilde{X}\right)_q \middle| \, \triangle_L(f) = 1 \otimes f \right\}.$$

Here $\triangle_L$ is the coaction $\triangle_L : \mathrm{Pol}\left(\widetilde{X}\right)_q \to \mathbb{C}[\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})]_q \otimes \mathrm{Pol}\left(\widetilde{X}\right)_q$, $\triangle_L : t_{ij} \mapsto \sum_{k=1}^{N} \pi(t_{ik}) \otimes t_{kj}$, and $\pi : \mathrm{Pol}\left(\widetilde{X}\right)_q \to \mathbb{C}[\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})]_q$ is the factorization map with respect to the two-sided ideal in $\mathrm{Pol}\left(\widetilde{X}\right)_q$ generated by t_{1k}, t_{k1}, $k=2,3,\ldots,N$, cf. [6, 11.6.2, 11.6.4].

2. Another characterization is in observing that $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is the subalgebra of $U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})$-invariants under the left action in $\mathrm{Pol}\left(\widetilde{X}\right)_q$. The latter action is a dual to the coaction $\triangle_L$ as in [6, 1.3.5, Proposition 15]. To prove the equivalence one should observe the $U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})$-invariance of $t_{1j}t_{1k}^*$ and compare the dimensions of graded components of the algebras $\mathrm{Pol}\left(\widehat{\mathcal{H}}_{n,m}\right)_q$ and $\mathbb{C}[GL_N]_q^{U_q\mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{N-1})}$.

We use the notation t_j instead of t_{1j} for the generators of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$.

Let I_φ, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, be the $*$-automorphism of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ defined on the generators $\{t_j\}_{j=1,\dots,N}$ by

$$I_\varphi : t_j \mapsto e^{i\varphi} t_j. \tag{3.2}$$

Then one more description of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ is as follows:

$$\mathrm{Pol}(\mathcal{H}_{n,m})_q \stackrel{\mathrm{def}}{=} \left\{ f \in \mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q \middle| \; I_\varphi(f) = f \quad \text{for all} \quad \varphi \right\}.$$

At the end of this section we list explicit formulas for the action of $U_q\mathfrak{su}_{n,m}$ on $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$.

The action of $U_q\mathfrak{su}_{n,m}$ on $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ is described as follows:

$$\begin{aligned}
E_j t_i &= \begin{cases} q^{-1/2} t_{i-1}, & j+1 = i, \\ 0, & \text{otherwise}, \end{cases} \\
F_j t_i &= \begin{cases} q^{1/2} t_{i+1}, & j = i, \\ 0, & \text{otherwise}, \end{cases} \\
K_j^{\pm 1} t_i &= \begin{cases} q^{\pm 1} t_i, & j = i, \\ q^{\mp 1} t_i, & j+1 = i, \\ t_i, & \text{otherwise}, \end{cases}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
E_j t_i^* &= \begin{cases} -q^{-3/2} t_{i+1}^*, & j = i \;\&\; i \neq n, \\ q^{-3/2} t_{i+1}^*, & j = i \;\&\; i = n, \\ 0, & \text{otherwise}, \end{cases} \\
F_j t_i^* &= \begin{cases} -q^{3/2} t_{i-1}^*, & j+1 = i \;\&\; i \neq n+1, \\ q^{3/2} t_{i-1}^*, & j+1 = i \;\&\; i = n+1, \\ 0, & \text{otherwise}, \end{cases} \\
K_j^{\pm 1} t_i^* &= \begin{cases} q^{\mp 1} t_i^*, & j = i, \\ q^{\pm 1} t_i^*, & j+1 = i, \\ t_i, & \text{otherwise}. \end{cases}
\end{aligned} \tag{3.4}$$

4 A $*$-Algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ of finite functions

Let us construct a faithful $*$-representation T of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ in a pre-Hilbert space $\mathcal{H}$ (the method of constructing T is well known; see, for example, [11]).

The space $\mathcal{H}$ is a linear span of its orthonormal basis $\{e(i_1, i_2, \dots, i_{N-1}) | \, i_1, \dots, i_n \in -\mathbb{Z}_+; \; i_{n+1}, \dots, i_{N-1} \in \mathbb{N}\}$.

The $*$-representation T is a restriction to $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ of the $*$-representation of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ defined by

$$\begin{aligned} T(t_j)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(q^{2(i_j-1)}-1\right)^{1/2} e(i_1,\dots,i_j-1,\dots,i_{N-1}), \\ T(t_j^*)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(q^{2i_j}-1\right)^{1/2} e(i_1,\dots,i_j+1,\dots,i_{N-1}), \end{aligned} \tag{4.1}$$

for $j \le n$,

$$\begin{aligned} T(t_j)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(1-q^{2(i_j-1)}\right)^{1/2} e(i_1,\dots,i_j-1,\dots,i_{N-1}), \\ T(t_j^*)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{j-1} i_k} \cdot \left(1-q^{2i_j}\right)^{1/2} e(i_1,\dots,i_j+1,\dots,i_{N-1}), \end{aligned} \tag{4.2}$$

for $n < j < N$, and, finally,

$$\begin{aligned} T(t_N)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1,\dots,i_{N-1}), \\ T(t_N^*)e(i_1,\dots,i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1,\dots,i_{N-1}). \end{aligned} \tag{4.3}$$

Define the elements $\{x_j\}_{j=1,\dots,N}$ as follows:

$$x_j \stackrel{\mathrm{def}}{=} \begin{cases} \sum\limits_{k=j}^{N} t_k t_k^*, & j > n, \\ -\sum\limits_{k=j}^{n} t_k t_k^* + \sum\limits_{k=n+1}^{N} t_k t_k^*, & j \le n. \end{cases} \tag{4.4}$$

Obviously, $x_1 = 1$, $x_i x_j = x_j x_i$,

$$t_j x_k = \begin{cases} q^2 x_k t_j, & j < k, \\ x_k t_j, & j \ge k, \end{cases} \tag{4.5}$$

hence

$$t_j^* x_k = \begin{cases} q^{-2} x_k t_j^*, & j < k, \\ x_k t_j^*, & j \ge k. \end{cases} \tag{4.6}$$

The vectors $e(i_1,\dots,i_{N-1})$ are joint eigenvectors of the operators $T(x_j)$, $j = 1,2,\dots,N$:

$$\begin{aligned} & T(x_1) = I, \\ & T(x_j)e(i_1,\dots,i_{N-1}) = q^{2\sum\limits_{k=1}^{j-1} i_k} e(i_1,\dots,i_{N-1}). \end{aligned} \tag{4.7}$$

The joint spectrum of the pairwise commuting operators $T(x_j)$, $j = 1,2,\dots,N$, is

$$\mathfrak{M} = \left\{(x_1,\dots,x_N) \in \mathbb{R}^N \right| \\ \left. x_i/x_j \in q^{2\mathbb{Z}} \;\&\; 1 = x_1 \le x_2 \le \ldots \le x_{n+1} > x_{n+2} > \ldots > x_N > 0\right\}.$$

Proposition 4.1 *T is a faithful representation of* $\mathrm{Pol}(\mathcal{H}_{n,m})_q$.

Proof. It suffices to verify faithfulness of the (unrestricted) representation T of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$. It is quite obvious that an arbitrary element of $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)_q$ can be written as a finite sum

$$f = \sum_{(i_1,\ldots,i_N,j_1,\ldots,j_N):\ i_k j_k = 0} t_1^{i_1}\ldots t_n^{i_n} t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N} f_{IJ}(x_2,\ldots,x_N) t_N^{j_N}\ldots t_{n+1}^{j_{n+1}} t_n^{*j_n}\ldots t_1^{*j_1},$$

where $f_{IJ}(x_2,\ldots,x_N)$ are polynomials, $I=(i_1,\ldots,i_N)$, $J=(j_1,\ldots,j_N)$. It follows from the definition of T that every summand

$$t_1^{i_1}\ldots t_n^{i_n} t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N} f_{IJ}(x_2,\ldots,x_N) t_N^{j_N}\ldots t_{n+1}^{j_{n+1}} t_n^{*j_n}\ldots t_1^{*j_1}$$

takes a basis vector $e(k_1,\ldots,k_{N-1})$ to a scalar multiple of the basis vector $e(k_1+j_1-i_1,\ldots,k_n+j_n-i_n,k_{n+1}-j_{n+1}+i_{n+1},\ldots,k_{N-1}-j_{N-1}+i_{N-1})$. Moreover, the sets of indices $(k_1+j_1-i_1,\ldots,k_{N-1}-j_{N-1}+i_{N-1})$ of the image basis vectors are different for different monomials, provided the indices of the initial monomial $e(k_1,\ldots,k_{N-1})$ have modules large enough. Therefore, to prove our claim, it suffices to choose arbitrarily a summand of f and to find an initial basis vector $e(k_1,\ldots,k_{N-1})$ in such a way that the chosen summand does not annihilate (under T) the vector $e(k_1,\ldots,k_{N-1})$.

Let us consider a basis vector $e(k_1,\ldots,k_{N-1})$ with $|k_s|>j_s$ for all $s=1,\ldots,N-1$. Then

$$T\left(t_N^{j_N}\ldots t_{n+1}^{j_{n+1}} t_n^{*j_n}\ldots t_1^{*j_1}\right) e(k_1,\ldots,k_{N-1}) = \\ \mathrm{const}\cdot e(k_1+j_1,\ldots,k_n+j_n,k_{n+1}-j_{n+1},\ldots,k_{N-1}-j_{N-1}),$$

where $\mathrm{const}\neq 0$.

Moreover, $T(f_{IJ}(x_2,\ldots,x_N))$ acts by multiplying the basis vector by a (value of a) polynomial $p\left(q^{2k_1},\ldots,q^{2k_{N-1}}\right)$ (due to (4.7)), where $p(u_1,u_2,\ldots,u_{N-1}) = f_{IJ}(u_1,u_1u_2,\ldots,u_1u_2\cdots u_{N-1})$, and p is certainly a nonzero polynomial. A routine argument allows one to find $k_1,\ldots,k_{N-1}$ such that $|k_s|>j_s$ and $p\left(q^{2k_1},\ldots,q^{2k_{N-1}}\right)\neq 0$. This proves the claim we need. $\square$

Let P be the orthogonal projection of $\mathcal{H}$ onto the linear span of vectors $\{e(\underbrace{0,\ldots,0}_{n},i_{n+1},\ldots,i_{N-1})|\,i_{n+1},\ldots,i_{N-1}\in\mathbb{N}\}$. Of course $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ does not contain an element f_0 such that $T(f_0)=P$. Our immediate intention is to add f_0 with this property.

Consider the $*$-algebra $\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right)\supset\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ derived from $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ by adding an element f_0 to its list of generators and the relations as below to its list of relations:

$$\begin{aligned} & t_j^* f_0 = f_0 t_j = 0, \qquad j\le n,\\ & x_{n+1}f_0 = f_0 x_{n+1} = f_0,\\ & f_0^2 = f_0^* = f_0,\\ & t_j f_0 = f_0 t_j;\qquad t_j^* f_0 = f_0 t_j^*,\qquad j\ge n+1. \end{aligned} \tag{4.8}$$

The relation $I_\varphi f_0 = f_0$ allows one to extend the $*$-automorphism I_φ (3.2) of the algebra $\mathrm{Pol}\left(\widetilde{\mathcal{H}}_{n,m}\right)$ to the $*$-automorphism of $\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right)$. Let

$$\mathrm{Fun}(\mathcal{H}_{n,m}) \stackrel{\mathrm{def}}{=} \left\{ f\in\mathrm{Fun}\left(\widetilde{\mathcal{H}}_{n,m}\right) \middle|\ I_\varphi f = f\right\}.$$

Obviously, there exists a unique extension of the $*$-representation T to a $*$-representation of the $*$-algebra $\mathrm{Fun}(\mathcal{H}_{n,m})$ such that $T(f_0)=P$.

Our subsequent observations involve extensively the two-sided ideal $\mathcal{D}(\mathcal{H}_{n,m})_q$ of $\mathrm{Fun}(\mathcal{H}_{n,m})$ generated by f_0. We call this ideal the algebra of finite functions on the quantum hyperbolic space.

Theorem 4.2 *The representation T of $\mathcal{D}(\mathcal{H}_{n,m})_q$ is faithful.*

Proof. Obviously, every $f\in\mathcal{D}(\mathcal{H}_{n,m})_q$ admits a unique decomposition

$$f=\sum_{\substack{(i_1\ldots,i_N,j_1\ldots j_N):\\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N=\\ =j_1+\ldots+j_n+i_{n+1}+\ldots+i_N}} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_0t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}.$$

A straightforward application of the commutation relations (4.8) allows us to refine the above decomposition as follows:

$$f=\sum_{\substack{(i_1\ldots,i_N,j_1\ldots j_N):\ i_kj_k=0\ \&\\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N=\\ =j_1+\ldots+j_n+i_{n+1}+\ldots+i_N}} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}, \tag{4.9}$$

where

$$f_{IJ}=\sum_K p_K(x_{n+2},\ldots,x_{N-1})t_1^{k_1}t_2^{k_2}\ldots t_n^{k_n}f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1} \tag{4.10}$$

for some nonzero polynomials p_K.

Let us consider a basis vector $e(a_1,\ldots,a_{N-1})$. Every summand from (4.9) takes $e(a_1,\ldots,a_{N-1})$ to a scalar multiple of the vector $e(a_1+j_1-i_1,\ldots,a_n+j_n-i_n,a_{n+1}-j_{n+1}+i_{n+1},\ldots,a_{N-1}-j_{N-1}+i_{N-1})$ (nonzero if well defined). By our assumptions on entries of I and J, the subset of nonzero multiples as above are linearly independent. Thus it suffices to choose arbitrarily a summand in (4.9) and to prove that it does not annihilate some basis vector.

Let us also choose arbitrarily a summand

$$p_K(x_{n+2},\ldots,x_{N-1})t_1^{k_1}t_2^{k_2}\ldots t_n^{k_n}f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1}$$

from (4.10). Now $T(f_0(t_n^*)^{k_n}\ldots(t_2^*)^{k_2}(t_1^*)^{k_1})T(t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1})e(a_1,\ldots,a_{N-1})=\mathrm{const}\cdot e(0,\ldots,0,a_{n+1}-j_{n+1},\ldots,a_{N-1}-j_{N-1})$. Here $\mathrm{const}=0$ unless $a_s+k_s+j_s=0$ for $s=1,\ldots,n$ and $a_s>j_s$ for $s=n+1,\ldots,N-1$. Set $a_s=-k_s-j_s$ for $s=1,\ldots,n$.

Now let us consider the action of $T(p_K(x_{n+2},\ldots,x_{N-1}))$ on vectors of the form $e(-k_1,\ldots,-k_n,a_{n+1}-j_{n+1},\ldots,a_{N-1}-j_{N-1})$ with $a_s>j_s$ for $s=n+1,\ldots,N-1$. An argument similar to that used in the final paragraph of the proof of Proposition 4.1 allows us to choose $a_{n+1},\ldots,a_{N-1}$ in such a way that $T\left(t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N}f_{IJ}t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1}\right)$ does not annihilate $e(a_1,\ldots,a_{N-1})$. This proves our claim. $\square$

R e m a r k 4.3 i) Due to (4.8), f_0 can be treated as a function of x_{n+1}:

$$f_0 = f_0(x_{n+1}) = \begin{cases} 1, & x_{n+1} = 1, \\ 0, & x_{n+1} \in q^{-2\mathbb{N}}. \end{cases} \tag{4.11}$$

(Recall that $\operatorname{spec} x_{n+1} = q^{-2\mathbb{Z}_+}$). Thus f_0 is a q-analog of the characteristic function of the submanifold

$$\left\{ (t_1, \ldots, t_N) \in \mathbb{C}^N \middle| \; t_1 = t_2 = \ldots = t_n = 0 \right\} \cap \mathcal{H}_{n,m}.$$

ii) Let $f(x_{n+1})$ be a polynomial. Then it follows from (4.4), (4.5) that

$$\sum_{i=1}^{n} t_i f(x_{n+1}) t_i^* = f(q^2 x_{n+1}) \sum_{i=1}^{n} t_i t_i^* = f\left(q^2 x_{n+1}\right)(x_{n+1} - 1). \tag{4.12}$$

This computation, together with (4.11), allows one to consider the element $f_1 = \sum\limits_{i=1}^{n} t_i f_0 t_i^*$ as a function of x_{n+1} such that

$$f_1(x_{n+1}) = \begin{cases} q^{-2} - 1, & x_{n+1} = q^{-2}, \\ 0, & x_{n+1} = 1 \;\text{ or }\; x_{n+1} \in q^{-2\mathbb{N}-2}. \end{cases}$$

Thus a multiple application of (4.12) leads to the following claim: $\mathcal{D}(\mathcal{H}_{n,m})_q$ contains *all* finite functions of x_{n+1} (i.e., such functions f that $f(q^{-n}) = 0$ for all but finitely many $n \in \mathbb{N}$).

Let us now is to endow $\mathcal{D}(\mathcal{H}_{n,m})_q$ with a structure of $U_q\mathfrak{su}_{n,m}$-module algebra. For that, it suffices to describe the action of the operators $\{E_j, F_j, K_j\}_{j=1,\ldots,N-1}$ on f_0. Here it is:

$$E_n f_0 = -\frac{q^{-1/2}}{q^{-2} - 1} t_n f_0 t_{n+1}^*, \tag{4.13}$$

$$F_n f_0 = -\frac{q^{3/2}}{q^{-2} - 1} t_{n+1} f_0 t_n^*, \tag{4.14}$$

$$K_n f_0 = f_0, \tag{4.15}$$

$$E_j f_0 = F_j f_0 = (K_j - 1) f_0 = 0, \qquad j \neq n. \tag{4.16}$$

R e m a r k 4.4 To see that the above structure of $U_q\mathfrak{su}_{n,m}$-module algebra on $\mathcal{D}(\mathcal{H}_{n,m})_q$ is well-defined, it suffices to use an argument contained in [11]. Here we restrict ourselves to explaining the motives which lead to (4.13) – (4.16). An application of (3.3), (3.4) and (4.4) allows one to conclude that for any polynomial $f(t)$

$$E_n f(x_{n+1}) = q^{-1/2} t_n \frac{f(q^{-2} x_{n+1}) - f(x_{n+1})}{q^{-2} x_{n+1} - x_{n+1}} t_{n+1}^*, \tag{4.17}$$

$$F_n f(x_{n+1}) = q^{3/2} t_{n+1} \frac{f(q^{-2} x_{n+1}) - f(x_{n+1})}{q^{-2} x_{n+1} - x_{n+1}} t_n^*, \tag{4.18}$$

$$E_j f = F_j f = (K_j - 1) f = 0 \;\text{ for }\; j \neq n,\; j = 1, 2, \ldots, N-1. \tag{4.19}$$

A subsequent application of (4.17) – (4.19) to the non-polynomial function f_0 (4.11) yields (4.13) – (4.16).

5 Invariant integral

The aim of this section is to present an explicit formula for a positive invariant integral on the space of finite functions $\mathcal{D}(\mathcal{H}_{n,m})_q$ and thereby to establish its existence.

Let $\nu_q : \mathcal{D}(\mathcal{H}_{n,m})_q \to \mathbb{C}$ be a linear functional defined by

$$\nu_q(f) = \mathrm{Tr}(T(f) \cdot Q) = \int\limits_{\mathcal{H}_{n,m}} f d\nu_q, \tag{5.1}$$

where $Q : \mathcal{H} \to \mathcal{H}$ is the linear operator given on the basis elements $e(i_1, \ldots, i_{N-1})$ by

$$Qe(i_1, \ldots, i_{N-1}) = \mathrm{const} \cdot q^{2\sum\limits_{j=1}^{N-1}(N-j)i_j} e(i_1, \ldots, i_{N-1}), \qquad \mathrm{const} > 0. \tag{5.2}$$

Thus $Q = \mathrm{const} \cdot T(x_2 \cdot \ldots \cdot x_N)$; this follows from (4.7).

Theorem 5.1 *The functional* ν_q *determined by* (5.1) *is well defined, positive, and* $U_q\mathfrak{su}_{n,m}$*-invariant.*

Proof. It follows from (3.1), (4.4), (4.5) that any element f of the algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ can be written in a unique way in the form

$$f = \sum_{\substack{(i_1 \ldots, i_N, j_1 \ldots j_N):\ i_k j_k = 0\ \& \\ i_1 + \ldots + i_n + j_{n+1} + \ldots + j_N = \\ = j_1 + \ldots + j_n + i_{n+1} + \ldots + i_N}} t_1^{i_1} \ldots t_n^{i_n} t_{n+1}^{*i_{n+1}} \ldots t_N^{*i_N} f_{IJ}(x_2, \ldots, x_N) t_N^{j_N} \ldots t_{n+1}^{j_{n+1}} t_n^{*j_n} \ldots t_1^{*j_1}, \tag{5.3}$$

with $f_{IJ}(x_2, \ldots, x_N)$ being a polynomial in $x_2, \ldots, x_n, x_{n+2}, \ldots, x_N$ and a *finite* function in x_{n+1}, that is, $f_{IJ}(x_2, \ldots, x_N)$ has the form

$$\sum_{\text{finite sum}} \alpha_{\mathbb{K}} x_2^{k_2} \cdots x_n^{k_n} f_{\mathbb{K}}(x_{n+1}) x_{n+2}^{k_{n+2}} \cdots x_N^{k_N}, \qquad \alpha_{\mathbb{K}} \in \mathbb{C}, \tag{5.4}$$

and $f_{\mathbb{K}}(q^{-2l}) \neq 0$ for finitely many $l \in \mathbb{Z}_+$.

Then, by our definition,

$$\nu_q : f \mapsto \mathrm{const} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \ldots, q^{2i_1+\ldots+2i_{N-1}}\right) \cdot$$

$$\cdot\, q^{2(N-1)i_1+2(N-2)i_2+\ldots+2i_{N-1}}, \tag{5.5}$$

and the series in the right hand side of (5.5) converges for f of the form (5.4).

The positivity of the linear functional ν_q means that

$$\nu_q(f^* f) > 0 \qquad \text{for} \quad f \neq 0.$$

This follows from the explicit formula (5.5) and the *faithfulness* of the $*$-representation T of the algebra $\mathcal{D}(\mathcal{H}_{n,m})_q$ (see Section 4).

What remains is to establish the $U_q\mathfrak{su}_{n,m}$-invariance of ν_q. The desired invariance is equivalent to

$$\nu_q(E_j f) = 0, \qquad \nu_q(F_j f) = 0. \tag{5.6}$$

for any $f \in \mathcal{D}(\mathcal{H}_{n,m})_q$ and $j = 1, 2, \ldots, N-1$. Observe that ν_q is a real functional, i.e., $\nu_q(f^*) = \overline{\nu_q(f)}$. The latter relation follows from selfadjointness of the operator $Q : \mathcal{H} \to \mathcal{H}$ involved in the definition of ν_q. This allows us to reduce the proof of (5.6) to proving the abridged version of it

$$\nu_q(E_j f) = 0, \qquad j = 1, 2, \ldots, N-1. \tag{5.7}$$

We are going to establish (5.7) for $j \le n$; for other j the proof is similar.

Moreover, for a function f of the form

$$f = t_1^{i_1} \ldots t_n^{i_n} t_{n+1}^{*i_{n+1}} \ldots t_N^{*i_N} f_{IJ}(x_2, \ldots, x_N) t_N^{j_N} \ldots t_{n+1}^{j_{n+1}} t_n^{*j_n} \ldots t_1^{*j_1}$$

with $i_k j_k = 0$ for $k = 1, 2, \ldots, N$, one has $\nu_q(E_j f) = 0$ if $I \ne (0, \ldots, \underset{(j+1)\text{th place}}{0, 1, 0}, \ldots, 0)$ and $J \ne (0, \ldots, \underset{j\text{th place}}{0, 1, 0}, \ldots, 0)$ (if $j < n$) or $I \ne (0, 0, \ldots, 0)$ and $J \ne (0, \ldots, \underset{j\text{th}\,(j+1)\text{th places}}{0, 1, 1, 0}, \ldots, 0)$ (if $j = n$). Thus we have to verify that $\nu_q\left(E_j\left(t_{j+1} f(x_2, \ldots, x_N) t_j^*\right)\right) = 0$.

It can be demonstrated by a direct computation that for $j \le n$

$$\begin{aligned} E_j(t_{j+1} f(x_2, \ldots, x_N) t_j^*) &= \\ = q^{-1/2} \Bigg[& q^2 f(x_2, \ldots, x_j, q^2 x_{j+1}, \ldots, q^2 x_N)(x_{j+1} - x_j) \frac{q^{-2} x_{j+2} - x_{j+1}}{(1-q^2) x_{j+1}} \\ & - f(x_2, \ldots, x_{j+1}, q^2 x_{j+2}, \ldots, q^2 x_N)(x_{j+2} - x_{j+1}) \frac{q^{-2} x_{j+1} - x_j}{(1-q^2) x_{j+1}} \Bigg]. \end{aligned} \tag{5.8}$$

1. Let $j = n$. Then

$$\begin{aligned} & \nu_q\left(E_j\left(t_{j+1} f(x_2, \ldots, x_N) t_j^*\right)\right) = \\ & = \text{const'} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} \Bigg[f\left(q^{2i_1}, \ldots, q^{2i_1 + \ldots + 2i_{n-1}}, q^{2i_1 + \ldots + 2i_n + 2}, \ldots, q^{2i_1 + \ldots + 2i_{N-1} + 2}\right) \cdot \\ & \quad \cdot \frac{q^2\left(q^{2i_1 + \ldots + 2i_n} - q^{2i_1 + \ldots + 2i_{n-1}}\right)\left(q^{2i_1 + \ldots + 2i_{n+1} - 2} - q^{2i_1 + \ldots + 2i_n}\right)}{q^{2i_1 + \ldots + 2i_n}} - \\ & \quad - f\left(q^{2i_1}, \ldots, q^{2i_1 + \ldots + 2i_n}, q^{2i_1 + \ldots + 2i_{n+1} + 2}, \ldots, q^{2i_1 + \ldots + 2i_{N-1} + 2}\right) \cdot \\ & \quad \cdot \frac{\left(q^{2i_1 + \ldots + 2i_{n+1}} - q^{2i_1 + \ldots + 2i_n}\right)\left(q^{2i_1 + \ldots + 2i_n - 2} - q^{2i_1 + \ldots + 2i_{n-1}}\right)}{q^{2i_1 + \ldots + 2i_n}} \Bigg] q^{2(N-1)i_1 + \ldots + 2i_{N-1}} = \end{aligned}$$

$$\begin{aligned} & = \text{const'} \cdot \sum_{\substack{(i_1 \ldots, i_n) \in (-\mathbb{Z}_+)^n \\ (i_{n+1}, \ldots, i_{N-1}) \in \mathbb{N}^{m-1}}} \Bigg[f\left(q^{2i_1}, \ldots, q^{2i_1 + \ldots + 2i_{n-1}}, q^{2i_1 + \ldots + 2i_n + 2}, \ldots, q^{2i_1 + \ldots + 2i_{N-1} + 2}\right) \cdot \\ & \quad \cdot q^2\left(q^{2i_n} - 1\right)\left(q^{2i_{n+1} - 2} - 1\right) - \\ & \quad - f\left(q^{2i_1}, \ldots, q^{2i_1 + \ldots + 2i_n}, q^{2i_1 + \ldots + 2i_{n+1} + 2}, \ldots, q^{2i_1 + \ldots + 2i_{N-1} + 2}\right) \cdot \\ & \quad \cdot \left(q^{2i_{n+1}} - 1\right)\left(q^{2i_n - 2} - 1\right) \Bigg] q^{2i_1 + \ldots + 2i_{n-1}} q^{2(N-1)i_1 + \ldots + 2i_{N-1}}. \end{aligned}$$

Let us consider the inner sum (in i_n and i_{n+1}). For brevity, we denote $f\left(q^{2i_1},\ldots,q^{2i_1+\ldots+2i_{n-1}},q^{2i_1+\ldots+2i_n+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2}\right)$ by $\psi_{i_n+1,i_{n+1}}$.

$$\sum_{\substack{i\in-\mathbb{Z}_+\\ j\in\mathbb{N}}}\left[\psi_{i+1,j}\cdot q^2\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)-\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)\right]q^{2(N-n)i+2(N-n-1)j}=$$

$$=\sum_{i\in-\mathbb{Z}_+,\,j\in\mathbb{N}}\psi_{i+1,j}\cdot\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j+2}$$
$$-\sum_{i\in-\mathbb{Z}_+,\,j\in\mathbb{N}}\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$=q^{-2(N-n-1)}\sum_{i\leq 1,j\in\mathbb{N}}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$-q^{-2(N-n-1)}\sum_{i\in-\mathbb{Z}_+,j\geq 2}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n)j}=0.$$

Thus the proof in this case is complete.

2. Let $j<n$.

$$\sum_{i,j\in-\mathbb{Z}_+}\left[\psi_{i+1,j}\cdot q^2\left(1-q^{2i}\right)\left(1-q^{2j-2}\right)-\psi_{i,j+1}\cdot\left(1-q^{2i-2}\right)\left(1-q^{2j}\right)\right]q^{2(N-n)i+2(N-n-1)j}=$$
$$=q^{-2(N-n-1)}\sum_{i\leq 1,j\in-\mathbb{Z}_+}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n-1)j}$$
$$-q^{-2(N-n-1)}\sum_{i\in-\mathbb{Z}_+,j\leq 1}\psi_{i,j}\left(1-q^{2i-2}\right)\left(1-q^{2j-2}\right)q^{2(N-n)i+2(N-n)j}=0.$$

The Theorem is proved. □

R e m a r k 5.2 It is reasonable to choose const in (5.2) so that the following normalization property is valid:

$$\nu_q(f_0)=1.$$

This allows us to find the constant explicitly:

$$\mathrm{const}=q^{-(N-n)(N-n-1)}\prod_{j=n+1}^{N-1}\left(1-q^{2(N-j)}\right).$$

6 Quantum homogeneous space $\Xi_{n,m}$

Let $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ denotes the quotient algebra of $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q$ by the ideal $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q\cdot c$ (recall that c belongs to the center of $\mathrm{Pol}(\widehat{\mathcal{H}}_{n,m})_q$). This is a q-analog of the polynomial algebra on the isotropic cone. Define an automorphism I_φ, $\varphi\in\mathbb{R}/2\pi\mathbb{Z}$, of the algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ by

$$I_\varphi(t_j)=e^{i\varphi}t_j,\qquad I_\varphi(t_j^*)=e^{-i\varphi}t_j^*.$$

Then it follows from the definition that

$$\mathrm{Pol}(\Xi_{n,m})_q = \left\{ f \in \mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q \middle|\ I_\varphi(f) = f \quad \text{for any} \quad \varphi \right\}.$$

We are going to construct a $*$-representation T_0 of the $*$-algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ in a pre-Hilbert space $\mathcal{H}_0$ in such a way that the restriction of T_0 to the subalgebra $\mathrm{Pol}(\Xi_{n,m})_q$ is a faithful $*$-representation of $\mathrm{Pol}(\Xi_{n,m})_q$.

Let $\{e(i_1, i_2, \ldots, i_{N-1}) |\ i_1 \in \mathbb{Z}; i_2, \ldots, i_n \in -\mathbb{Z}_+;\ i_{n+1}, \ldots, i_{N-1} \in \mathbb{N}\}$ be the orthonormal basis of the space $\mathcal{H}_0$. Then T_0 is defined as follows.

$$\begin{aligned} T_0(t_1)e(i_1, \ldots, i_{N-1}) &= q^{i_1-1}e(i_1 - 1, \ldots, i_{N-1}), \\ T_0(t_1^*)e(i_1, \ldots, i_{N-1}) &= q^{i_1}e(i_1 + 1, \ldots, i_{N-1}), \end{aligned} \tag{6.1}$$

$$\begin{cases} T_0(t_j)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(q^{2(i_j-1)} - 1\right)^{1/2} e(i_1, \ldots, i_j - 1, \ldots, i_{N-1}), \\ T_0(t_j^*)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(q^{2i_j} - 1\right)^{1/2} e(i_1, \ldots, i_j + 1, \ldots, i_{N-1}), \\ \qquad \text{for} \quad 1 < j \le n, \end{cases} \tag{6.2}$$

$$\begin{cases} T_0(t_j)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(1 - q^{2(i_j-1)}\right)^{1/2} e(i_1, \ldots, i_j - 1, \ldots, i_{N-1}), \\ T_0(t_j^*)e(i_1, \ldots, i_{N-1}) = q^{\sum\limits_{k=1}^{j-1} i_k} \left(1 - q^{2i_j}\right)^{1/2} e(i_1, \ldots, i_j + 1, \ldots, i_{N-1}), \\ \qquad \text{for} \quad n < j < N, \end{cases} \tag{6.3}$$

$$\begin{aligned} T_0(t_N)e(i_1, \ldots, i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1, \ldots, i_{N-1}), \\ T_0(t_N^*)e(i_1, \ldots, i_{N-1}) &= q^{\sum\limits_{k=1}^{N-1} i_k} e(i_1, \ldots, i_{N-1}), \end{aligned} \tag{6.4}$$

Let us introduce the notation

$$\xi_j = \begin{cases} \sum\limits_{k=j}^{N} t_k t_k^*, & j > n, \\ -\sum\limits_{k=j}^{n} t_k t_k^* + \sum\limits_{k=n+1}^{N} t_k t_k^*, & j \le n. \end{cases}$$

Evidently, $\xi_1 = 0$, and the elements $\xi_2, \ldots, \xi_N$ satisfy (4.5) – (4.6) with x_k being replaced by ξ_k. The joint spectrum of the pairwise commuting operators $\{T_0(\xi_j)\}_{j=\overline{1,N}}$ is the set

$$\mathfrak{M}_0 = \big\{(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \big| \\ \xi_j \in q^{2\mathbb{Z}},\ j > 1\ \&\ 0 = \xi_1 \le \xi_2 \le \ldots \le \xi_{n+1} > \xi_{n+2} > \ldots > \xi_N > 0\big\}.$$

Similarly to the case of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$, any element from $\mathrm{Pol}(\Xi_{n,m})_q$ can be written in the form

$$f = \sum_{\substack{IJ=0 \\ \text{finite sum} \\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N= \\ =i_{n+1}+\ldots+i_N+j_1+\ldots+j_n}} t_1^{i_1}\ldots t_n^{i_n}t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N} f_{IJ}(\xi_2,\ldots,\xi_N) t_N^{j_N}\ldots t_{n+1}^{j_{n+1}}t_n^{*j_n}\ldots t_1^{*j_1},$$

where f_{IJ} are polynomials in $\xi_2,\ldots,\xi_N$, and such decomposition is unique.

The $*$-algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ is a $U_q\mathfrak{su}_{n,m}$-module algebra. Namely, the action of $U_q\mathfrak{su}_{n,m}$ on the generators t_j, t_j^* of $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ is defined by (3.3) – (3.4). This definition is correct due to the fact that the element c of the covariant algebra $\mathrm{Pol}(\widetilde{\mathcal{H}}_{n,m})_q$ is $U_q\mathfrak{su}_{n,m}$-invariant. Thus the $*$-algebra $\mathrm{Pol}(\Xi_{n,m})_q$ is a $U_q\mathfrak{su}_{n,m}$-module algebra too. The same computations as in the case of $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ show that for any polynomial $f(t)$

$$\begin{aligned} E_n f(\xi_{n+1}) &= q^{-1/2}t_n\frac{f(q^{-2}\xi_{n+1})-f(\xi_{n+1})}{q^{-2}\xi_{n+1}-\xi_{n+1}}t_{n+1}^*, \\ F_n f(\xi_{n+1}) &= q^{3/2}t_{n+1}\frac{f(q^{-2}\xi_{n+1})-f(\xi_{n+1})}{q^{-2}\xi_{n+1}-\xi_{n+1}}t_n^*, \\ (K_n-1)f(\xi_{n+1}) &= E_jf(\xi_{n+1}) = F_jf(\xi_{n+1}) = (K_j-1)f(\xi_{n+1}) = 0, \quad j\neq n. \end{aligned} \tag{6.5}$$

Now (4.5), (4.6), and (6.5) allow one to introduce the covariant $*$-algebra $\mathcal{D}(\Xi_{n,m})$ of finite functions on the quantum homogeneous space $\Xi_{n,m}$. It is formed by elements of the form (5.3) with ξ_k instead of x_k, where $f_{IJ}(\xi_2,\ldots,\xi_N)$ are polynomials of $\xi_2,\ldots,\xi_n,\xi_{n+2},\ldots,\xi_N$ and finite functions of ξ_{n+1} (i.e., f_{IJ} has the form (5.4) where $f_{\mathbb{K}}(q^{2l})\neq 0$ for finitely many $l\in\mathbb{Z}$).

Theorem 6.1 *T_0 can be extended to a faithful $*$-representation of the $*$-algebra $\mathcal{D}(\Xi_{n,m})$.*

R e m a r k 6.2 The algebra $\mathrm{Pol}(\mathcal{H}_{n,m})_q$ has the same list of generators as $\mathrm{Pol}(\widetilde{\Xi})_q$ while the lists of relations differ by replacing $c-1=0$ with $c=0$. Furthermore, the differences between the formulas (4.1) – (4.3) and (6.1) – (6.4) are low enough to enable us to apply the same argument in proving Theorems 6.1 and 4.2.

Now let us construct an invariant integral on $\mathcal{D}(\Xi_{n,m})$. Denote by ν_q^0 the linear functional $\nu_q^0:\mathcal{D}(\Xi_{n,m})\to\mathbb{C}$ given by

$$\nu_q^0(f) = \mathrm{Tr}(T_0(f)\cdot Q_0)\ \left(=\int_{\Xi_{n,m}} f d\nu_q^0\right) \tag{6.6}$$

with $Q_0:\mathcal{H}_0\to\mathcal{H}_0$ being the linear map given by

$$Q_0 e(i_1,\ldots,i_{N-1}) = \mathrm{const}\cdot q^{2\sum\limits_{j=1}^{N-1}(N-j)i_j} e(i_1,\ldots,i_{N-1}). \tag{6.7}$$

Theorem 6.3 *The functional ν_q^0 is well-defined, positive, and $U_q\mathfrak{su}_{n,m}$-invariant.*

Proof. It follows from the definition that

$$\nu_q^0(f) = \text{const} \cdot \sum_{\substack{i_1 \in \mathbb{Z} \\ (i_2 \dots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \dots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \dots, q^{2i_1+\dots+2i_{N-1}}\right) q^{2i_1(N-1)+\dots+2i_{N-1}}. \tag{6.8}$$

Here f_{00} is the function involved in the decomposition (5.3) of f.

To prove that the definition (6.6) of ν_q^0 is correct, it now suffices to show that the series in the r.h.s. of (6.8) is absolutely convergent for f_{00} satisfying the condition

$$f_{00}\left(\xi_2, \dots, \xi_n, q^{2l}, \xi_{n+2}, \dots, \xi_N\right) = 0 \quad \text{for } l \neq l_0.$$

Let f_{00} be such a function. Then

$$\sum_{\substack{i_1 \in \mathbb{Z} \\ (i_2 \dots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \dots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2i_1}, q^{2i_1+2i_2}, \dots, q^{2i_1+\dots+2i_n} q^{2i_1+\dots+2i_{N-1}}\right) q^{2i_1(N-1)+\dots+2i_{N-1}} =$$

$$= \sum_{\substack{(i_2 \dots, i_n) \in (-\mathbb{Z}_+)^{n-1} \\ (i_{n+1}, \dots, i_{N-1}) \in \mathbb{N}^{m-1}}} f_{00}\left(q^{2l_0-2i_2-\dots-2i_n}, q^{2l_0-2i_3-\dots-2i_n}, \dots, q^{2l_0-2i_n}, q^{2l_0}, q^{2l_0+2i_{n+1}}, \dots\right) \cdot$$

$$\cdot\, q^{2l_0(N-1)} \cdot q^{2i_1(N-1)+\dots+2i_{N-1}} \cdot q^{-2i_2-4i_3-\dots-2(n-1)i_n} \cdot q^{2i_{n+1}(m-1)+i_{n+2}(m-2)+\dots+2i_{N-1}}. \tag{6.9}$$

It is implicit here that only terms with $i_1+\dots+i_n = l_0$ can be non-zero; also, the following obvious equality is used:

$$q^{2(N-1)i_1+\dots+2i_{N-1}} = q^{2i_1} \cdot q^{2i_1+2i_2} \cdot \dots \cdot q^{2i_1+\dots+2i_{N-1}}.$$

Now to establish the convergence of the series (6.9), it suffices to recall that f_{00} is a polynomial in $\xi_2, \dots, \xi_n, \xi_{n+2}, \dots, \xi_N$.

The positive definiteness of ν_q^0 can be explained in the same way as it was done in Section 5 for ν_q.

Let us turn to proving the invariance of ν_q^0. To do this, one needs to reproduce the proof of a similar fact for ν_q almost literally, including the computations of cases 1 and 2. But now there is one more case to be considered:

3. Let $j = 1$, then (see (5.8))

$$E_1(t_2 f(\xi_2, \dots, \xi_N) t_1^*) =$$
$$= q^{-1/2}\left[f(q^2\xi_2, \dots, q^2\xi_N)\frac{\xi_2(\xi_3 - q^2\xi_2)}{(1-q^2)\xi_2} - f(\xi_2, q^2\xi_3, \dots, q^2\xi_N)\frac{q^{-2}\xi_2(\xi_3-\xi_2)}{(1-q^2)\xi_2}\right] =$$
$$= \frac{q^{-1/2}}{1-q^2}\left[f(q^2\xi_2, \dots, q^2\xi_N)(\xi_3 - q^2\xi_2) - q^{-2} f(\xi_2, q^2\xi_3, \dots, q^2\xi_N)(\xi_3 - \xi_2)\right].$$

Now let us show that $\nu_q^0(E_1(t_2 f(\xi_2,\ldots,\xi_N)t_1^*))=0$. In fact,

$$\begin{aligned}&\nu_q^0(E_1(t_2 f(\xi_2,\ldots,\xi_N)t_1^*))=\\&=\mathrm{const}'\cdot\sum_{\substack{i_1\in\mathbb{Z}\\(i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1}\\(i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}}\Big[f(q^{2i_1+2},q^{2i_1+2i_2+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})(q^{2i_2-2}-1)q^{2i_1+2}\\&\qquad-f(q^{2i_1},q^{2i_1+2i_1+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})q^{-2}(q^{2i_2}-1)q^{2i_1}\Big]\,q^{2i_1(N-1)+\ldots+2i_{N-1}}.\end{aligned}\tag{6.10}$$

As usual, we denote $f(q^{2i_1+2},q^{2i_1+2i_2+2},\ldots,q^{2i_1+\ldots+2i_{N-1}+2})$ by ψ_{i_1+1,i_2}. Let us compute the inner sum over i_1 and i_2 in the r.h.s. of (6.10).

$$\begin{aligned}&\sum_{i\in\mathbb{Z},j\in-\mathbb{Z}_+}\left[q^2\psi_{i+1,j}(q^{2j-2}-1)-q^{-2}\psi_{i,j+1}(q^{2j}-1)\right]\cdot q^{2iN}q^{2j(N-2)}=\\&=\sum_{i\in\mathbb{Z},j\in-\mathbb{Z}_+}\psi_{i,j}(q^{2j-2}-1)\cdot q^{2iN+2jN-4j-2N+2}-\sum_{i\in\mathbb{Z},j\leq 1}\psi_{i,j}(q^{2j-2}-1)\cdot q^{2iN+2j(N-2)-2N+2}=0.\ \Box\end{aligned}$$

R e m a r k 6.4 Here const is chosen in (6.7) so that the following normalization property is valid:

$$\nu_q^0(f_0)=1.$$

This allows us to find the constant explicitly:

$$\mathrm{const}=q^{-(N-n)(N-n-1)}\prod_{j=1}^{n-1}\left(1-q^{2j}\right)\prod_{j=1}^{N-n-1}\left(1-q^{2j}\right).$$

7 Principal non-unitary and unitary series of representations of $U_q\mathfrak{su}_{n,m}$ related to the space $\Xi_{n,m}$

The element ξ_{n+1} quasi-commutes with all the generators of the algebra $\mathrm{Pol}(\Xi_{n,m})_q$. Thus $(\xi_{n+1})^{\mathbb{Z}_+}$ is an Ore set and one can consider a localization $\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}$ of the algebra $\mathrm{Pol}(\Xi_{n,m})_q$ with respect to the multiplicative system $(\xi_{n+1})^{\mathbb{Z}_+}$. Evidently, the $U_q\mathfrak{su}_{n,m}$-module algebra structure extends to the localization in a unique way.

Denote by γ the automorphism of the algebra $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$ given on the generators by

$$\gamma: t_j\mapsto qt_j,\qquad t_j^*\mapsto qt_j^*.$$

Note that γ is well defined due to the homogeneity of the defining relations for $\mathrm{Pol}\left(\widetilde{\Xi}_{n,m}\right)_q$. Obviously, $\gamma(\xi_{n+1})=q^2\xi_{n+1}$, and this allows one to extend γ to an automorphism of the algebra $\mathrm{Pol}(\Xi_{n,m})_{q,\xi_{n+1}}$, which commutes with the action of $U_q\mathfrak{su}_{n,m}$. This can be deduced from (3.3), (3.4), and (6.5).

Introduce the $*$-algebra $\mathcal{E}(\Xi_{n,m})_q$ of elements of the form

$$f = \sum_{\substack{IJ=0 \\ \text{finite sum} \\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N= \\ =i_{n+1}+\ldots+i_N+j_1+\ldots+j_n}} t_1^{i_1}\ldots t_n^{i_n} t_{n+1}^{*i_{n+1}}\ldots t_N^{*i_N} f_{IJ}(\xi_2,\ldots,\xi_N) t_N^{j_N}\ldots t_{n+1}^{j_{n+1}} t_n^{*j_n}\ldots t_1^{*j_1},$$

with

$$f_{IJ}(\xi_2,\ldots,\xi_N) = \sum_{\substack{\text{finite sum} \\ k_2,\ldots,k_n,k_{n+2},\ldots,k_N\in\mathbb{Z}_+ \\ k_{n+1}\in\mathbb{C}}} \alpha_{\mathbb{K}} \xi_2^{k_2}\xi_3^{k_3}\ldots\xi_N^{k_N}. \tag{7.1}$$

Here $\alpha_{\mathbb{K}}\in\mathbb{C}$ and the algebra structure is given by (4.5), (4.6).

Given $s\in\mathbb{C}$, let $\mathcal{E}_s(\Xi_{n,m})_q$ be the subspace in $\mathcal{E}(\Xi_{n,m})_q$ of those elements which have the 'homogeneity degree' equal to $s-N+1$:

$$\gamma(f) = q^{s-N+1}\cdot f. \tag{7.2}$$

Thus $\mathcal{E}_s(\Xi_{n,m})_q$ is a $U_q\mathfrak{su}_{n,m}$-submodule in $\mathcal{E}(\Xi_{n,m})_q$. We call these submodules the modules of the principal non-unitary series related to $\Xi_{n,m}$.

Now let us construct an invariant integral in $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$.

Note that $\mathcal{D}(\Xi_{n,m})_q$ can be made a covariant $\mathcal{E}(\Xi_{n,m})_q$-bimodule using the relations (4.5), (4.6).

Let $\chi_l\in\mathcal{D}(\Xi_{n,m})_q$ be the function of ξ_{n+1} such that

$$\chi_l(q^{2k}) = \delta_{kl}, \qquad k,l\in\mathbb{Z}.$$

Lemma 7.1 *For any $f\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$, the integral*

$$b_q^{(l)}(f) \stackrel{\text{def}}{=} \int\limits_{\Xi_{n,m}} f\cdot\chi_l d\nu_q^0 \tag{7.3}$$

does not depend on l.

Proof.

$$\begin{aligned} b_q^{(l)}(f) &= \\ &= \text{const} \sum_{\substack{i_1\in\mathbb{Z} \\ (i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1} \\ (i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}} f_{00}(q^{2i_1},q^{2i_1+2i_2},\ldots,q^{2i_1+\ldots+2i_{N-1}})\chi_l(q^{2i_1+\ldots+2i_{N-1}})q^{2i_1(N-1)+\ldots+2i_{N-1}} = \\ &= \text{const} \sum_{\substack{(i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1} \\ (i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}} f_{00}(q^{2l-2i_2-\ldots-2i_n},q^{2l-2i_3-\ldots-2i_n},\ldots,q^{2l-2i_n},q^{2l},q^{2l+2i_{n+1}},\ldots)\cdot \\ &\qquad \cdot q^{2l(N-1)}\cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}}. \end{aligned} \tag{7.4}$$

Clearly, $f \in \mathcal{E}_{-N+1}(\Xi_{n,m})_q$ implies

$$\gamma(f_{00}(\xi_2,\ldots,\xi_N)) = q^{-2N+2} f_{00}(\xi_2,\ldots,\xi_N),$$

or, equivalently,

$$f_{00}(q^2\xi_2,\ldots,q^2\xi_N) = q^{-2N+2} f_{00}(\xi_2,\ldots,\xi_N),$$

and thus the r.h.s. of (7.4) can be rewritten as follows

$$\begin{gathered}\mathrm{const} \sum_{\substack{(i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1}\\ (i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}} q^{2l(N-1)} f_{00}(q^{-2i_2-\ldots-2i_n}, q^{-2i_3-\ldots-2i_n},\ldots,q^{-2i_n},1,q^{2i_{n+1}},\ldots)\cdot\\ \cdot q^{2l(N-1)}\cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}} =\\ = \mathrm{const} \sum_{\substack{(i_2\ldots,i_n)\in(-\mathbb{Z}_+)^{n-1}\\ (i_{n+1},\ldots,i_{N-1})\in\mathbb{N}^{m-1}}} f_{00}(q^{-2i_2-\ldots-2i_n}, q^{-2i_3-\ldots-2i_n},\ldots,q^{-2i_n},1,q^{2i_{n+1}},\ldots)\cdot\\ \cdot q^{-2i_2-4i_3-\ldots-2(n-1)i_n+2i_{n+1}(m-1)+2i_{n+2}(m-2)+\ldots+2i_{N-1}}. \;\square \end{gathered} \tag{7.5}$$

Introduce the notation $b_q(f)$ or $\int f db_q$ for the linear functional (7.3) on $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. It follows from the proof of Lemma 7.1 that

$$\begin{gathered} b_q(f) = (q^{-2}-1)^N\cdot\\ \cdot \sum_{\substack{(j_1\ldots,j_{n-1})\in(-\mathbb{Z}_+)^{n-1}\\ (i_1,\ldots,i_{m-1})\in\mathbb{N}^{m-1}}} f_{00}(q^{2j_1+\ldots+2j_{n-1}}, q^{2i_2+\ldots+2j_{n-1}},\ldots,q^{-2j_{n-1}},1,q^{2i_1},q^{2i_1+2i_2},\ldots,q^{2i_1+\ldots,2i_{m-1}})\cdot\\ \cdot q^{2j_1+4j_2+\ldots+2(n-1)j_{n-1}}\cdot q^{2(m-1)i_1+2(m-2)i_2+\ldots+2i_{m-1}}. \end{gathered} \tag{7.6}$$

Theorem 7.2 *b_q is an invariant integral on $\mathcal{E}_{-N+1}(\Xi_{n,m})_q$.*

Proof. By (6.5), the functions of ξ_{n+1} are $U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_m)$-invariants. Thus b_q is a $U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_m)$-invariant functional (see Theorem 6.3). It remains to prove that $b_q(F_nf) = b_q(E_nf) = 0$ for $f\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Let us prove just one of these two equalities, for example, $b_q(E_nf) = \int\limits_{\Xi_{n,m}} E_nf\cdot\chi_l d\nu_q^0 = 0$.

The invariance of ν_q^0 and the fact that $\mathcal{D}(\Xi_{n,m})_q$ is a covariant $\mathcal{E}(\Xi_{n,m})_q$-bimodule imply that

$$b_q(E_nf) = -q^{-1}\int f\cdot E_n\chi_l d\nu_q^0,\qquad f\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$$

(the integration by parts is used here, see [1, Chapter 4]).

By (6.5),

$$
\begin{aligned}
-q^{-1}\int f\cdot E_n\chi_l d\nu_q^0 &= -q^{-1}\int f\cdot q^{-1/2}t_n\frac{\chi_l(q^{-2}\xi_{n+1})-\chi_l(\xi_{n+1})}{(q^{-2}-1)\xi_{n+1}}t_{n+1}^* d\nu_q^0 = \\
&= -\frac{q^{-3/2}}{(q^{-2}-1)}\int f\cdot t_n\frac{\chi_{l+1}(\xi_{n+1})-\chi_l(\xi_{n+1})}{\xi_{n+1}}t_{n+1}^* d\nu_q^0 = \\
&= -\frac{q^{-3/2}}{(q^{-2}-1)}\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}t_{n+1}^*\right)Q_0\right] = \\
&= -\frac{q^{-3/2}}{(q^{-2}-1)}(q^{-2}-1)^N\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}t_{n+1}^*\xi_2\xi_3\ldots\xi_N\right)\right] = \\
&= \operatorname{const}(q,n,N)\operatorname{Tr}\left[T_0\left(f\cdot t_n\frac{\chi_{l+1}-\chi_l}{\xi_{n+1}}\xi_2\xi_3\ldots\xi_N t_{n+1}^*\right)\right] = \\
&= \operatorname{const}(q,n,N)\operatorname{Tr}\left[T_0\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)\xi_2\xi_3\ldots\xi_N\right)\right] = \\
&= \operatorname{const}'(q,n,N)\operatorname{Tr}\left[T_0\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)Q_0\right)\right] = \\
&= \operatorname{const}'(q,n,N)\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}(\chi_{l+1}-\chi_l)d\nu_1^0. \qquad (7.7)
\end{aligned}
$$

If $f\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$, one has $t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Thus the latter expression in (7.7) can be rewritten as follows:

$$
\begin{aligned}
\operatorname{const}'(q,n,N)&\left(\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\chi_{l+1}d\nu_1^0-\int t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\chi_l d\nu_1^0\right) = \\
&= \operatorname{const}'(q,n,N)\left(b_q^{(l+1)}\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\right)-b_q^l\left(t_{n+1}^*f\cdot t_n\frac{1}{\xi_{n+1}}\right)\right).
\end{aligned}
$$

It follows from Lemma 7.1 that the latter difference is zero. □

If $f_1\in\mathcal{E}_s(\Xi_{n,m})_q$ and $f_2\in\mathcal{E}_{-s}(\Xi_{n,m})_q$, one has $f_1\cdot f_2\in\mathcal{E}_{-N+1}(\Xi_{n,m})_q$. Now an application of the standard arguments (see, e.g., [1, Chapter 4]) which set correspondence between invariant integrals and invariant pairings, yields

Corollary 7.3 *The pairing $\mathcal{E}_s(\Xi_{n,m})_q\times\mathcal{E}_{-s}(\Xi_{n,m})_q\to\mathbb{C}$,*

$$(f_1,f_2)\mapsto\langle f_1,f_2\rangle\stackrel{\text{def}}{=}\int f_1f_2db_q$$

is $U_q\mathfrak{su}_{n,m}$-invariant.

Obviously, the involution $*$ of the $*$-algebra $\mathcal{E}(\Xi_{n,m})_q$ maps $\mathcal{E}_{i\lambda}(\Xi_{n,m})_q$ to $\mathcal{E}_{-i\lambda}(\Xi_{n,m})_q$ for $\lambda\in\mathbb{R}$.

Proposition 7.4 *The sesquilinear form*

$$(f_1,f_2)=\int f_2^*f_1db_q,\qquad f_1,f_2\in\mathcal{E}_{i\lambda}(\Xi_{n,m})_q, \tag{7.8}$$

is invariant and positive definite.

Proof. The invariance follows immediately from Corollary 7.3 (the standard arguments from [1, Chapter 4] are to be applied here again).

To see that the form (7.8) is positive definite, one should recall that the integral ν_q^0 is positive definite (Theorem 6.3), and use the following computations:

$$(f,f) = \int f^* f db_q = \int\limits_{\Xi_{n,m}} f^* f \chi_l d\nu_q^0 = \mathrm{Tr}\left(T_0\left(f^* f \chi_l\right) Q_0\right) = \mathrm{Tr}\left(T_0\left(f^* f \chi_l \chi_l\right) Q_0\right) =$$
$$= \mathrm{Tr}\left(T_0\left(f^* f \chi_l \cdot \mathrm{const} \cdot \xi_2 \ldots \xi_N \chi_l\right)\right) = \mathrm{Tr}\left(T_0\left(\chi_l f^* f \chi_l\right) Q_0\right) = \mathrm{Tr}\left(T_0\left(\chi_l^* f^* f \chi_l\right) Q_0\right) =$$
$$= \int\limits_{\Xi_{n,m}} (f\chi_l)^* f \chi_l d\nu_q^0.$$

Here $f \in \mathcal{E}_{i\lambda}(\Xi_{n,m})_q$, $\lambda \in \mathbb{Z}$, and the obvious relations $\chi_l^2 = \chi_l$, $\chi_l^* = \chi_l$, $\chi_l \xi_k = \xi_k \chi_l$ are used. □

Thus $\mathcal{E}_{i\lambda}(\Xi_{n,m})_q$, $\lambda \in \mathbb{R}$, are unitary $U_q\mathfrak{su}_{n,m}$-modules. They will be called the modules of the principal unitary series related to $\Xi_{n,m}$.

Let us look at the structure of $\mathcal{E}_\lambda(\Xi_{n,m})_q$ as a $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_m)$-module. Let $L^{(n)}(\lambda)$ be the finite dimensional simple $U_q\mathfrak{sl}_n$-module with highest weight λ. Also let ϖ_j, $j = 1, \ldots, n-1$, be the fundamental weights of the Lie algebra $\mathfrak{sl}_n$.

Now we recall that if A is a Hopf algebra and V_1 is an A-module, and B is a Hopf algebra and V_2 is a B-module then $V_1 \boxtimes V_2$ denotes their tensor product endowed with the structure of $A \otimes B$-module in the natural way.

Theorem 7.5 *The $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_m)$-module $\mathcal{E}_{2s}(\Xi_{n,m})_q$ splits as a multiplicity free direct sum of its simple submodules*

$$L^{(n)}(k\omega_1 + l\omega_{n-1}) \boxtimes L^{(m)}(l'\omega_1 + k'\omega_{m-1}), \tag{7.9}$$

with $k, l, k', l' \geq 0$, $k + l' = k' + l$. Every such submodule is generated by the highest weight vector of the form

$$t_1^k t_N^{*k'} \xi_{n+1}^{(s-k'-l)} t_{n+1}^{l'} t_n^{*l}. \tag{7.10}$$

Proof. For simplicity, we prove the claim in the special case $s = (N-1)/2$, the other cases are similar. Each element $f \in \mathcal{E}_{N-1}(\Xi_{n,m})_q$ can be decomposed in the following way

$$f = \sum_{\substack{\text{finite sum} \\ i_1+\ldots+i_n+j_{n+1}+\ldots+j_N= \\ =i_{n+1}+\ldots+i_N+j_1+\ldots+j_n=\lambda}} t_1^{i_1} \ldots t_n^{i_n} t_{n+1}^{*i_{n+1}} \ldots t_N^{*i_N} \cdot \xi_{n+1}^{-\lambda} \cdot t_N^{j_N} \ldots t_{n+1}^{j_{n+1}} t_n^{*j_n} \ldots t_1^{*j_1}.$$

Evidently, in all such decompositions $\lambda \in \mathbb{Z}_+$. For every fixed decomposition of f let us consider the largest λ through all the terms, and then denote by $\lambda(f)$ the smallest λ throughout all such decompositions of f. Now we introduce a filtration

$$\mathcal{E}_{N-1}(\Xi_{n,m})_q = \bigcup_{a=0}^{\infty} \mathcal{E}_{N-1}(\Xi_{n,m})_{q,a},$$

where

$$\mathcal{E}_{N-1}(\Xi_{n,m})_{q,a} = \{f \in \mathcal{E}_{N-1}(\Xi_{n,m})_q \mid \lambda(f) \leq a\}.$$

By obvious reasons, $t_1^k t_N^{*k'} \xi_{n+1}^{((N-1)/2-k'-l)} t_{n+1}^{l'} t_n^{*l}$ generates a $U_q(\mathfrak{sl}_\mathfrak{n} \times \mathfrak{sl}_\mathfrak{m})$-module isomorphic to $L^{(n)}(k\omega_1 + l\omega_{n-1}) \boxtimes L^{(m)}(l'\omega_1 + k'\omega_{m-1})$. Also, the $U_q(\mathfrak{sl}_\mathfrak{n} \times \mathfrak{sl}_\mathfrak{m})$-action does not increase $\lambda(f)$, so $L^{(n)}(k\omega_1 + l\omega_{n-1}) \boxtimes L^{(m)}(l'\omega_1 + k'\omega_{m-1}) \subset \mathcal{E}_{N-1}(\Xi_{n,m})_{q,a}$ if $k + l' \leq a$. The fact that a direct sum of all such modules exhaust $\mathcal{E}_{N-1}(\Xi_{n,m})_{q,a}$ can be obtained by calculating the dimensions. In fact, we have to verify that

$$\dim \mathcal{E}_{N-1}(\Xi_{n,m})_{q,a} \leq \sum_{k+l' \leq a} \dim \left(L^{(n)}(k\omega_1 + l\omega_{n-1}) \boxtimes L^{(m)}(l'\omega_1 + k'\omega_{m-1}) \right).$$

Since one has the relation $\xi_1 = 0$ in $\mathcal{E}(\Xi_{n,m})_q$, the dimension of $\mathcal{E}_{N-1}(\Xi_{n,m})_{q,a}$ satisfies the following inequality:

$$\dim \mathcal{E}_{N-1}(\Xi_{n,m})_{q,a} \leq (C_{a+N-1}^{N-1})^2.$$

It is sufficient to verify the inequality

$$(C_{a+N-1}^{N-1})^2 \leq \sum_{k+l' \leq a} \dim \left(L^{(n)}(k\omega_1 + l\omega_{n-1}) \boxtimes L^{(m)}(l'\omega_1 + k'\omega_{m-1}) \right)$$

in the classical case. In the classical context this can be obtained via an induction argument in a. □

We are going to establish the necessary conditions for $\mathcal{E}_s(\Xi_{n,m})_q$ to be equivalent as $U_q\mathfrak{sl}_N$-modules.

A special construction associates to every finite dimensional representation V of $U_q\mathfrak{sl}_N$ a central element C_V of some extended algebra $U_q^{\mathrm{ext}}\mathfrak{sl}_N \supset U_q\mathfrak{sl}_N$ [6]. It follows that the collection of constants $C_{L(\omega_p)}$, $p = 1, \ldots, N$, constitute an invariant of isomorphism for $\mathcal{E}_{2s}(\Xi_{n,m})_q$ as $U_q\mathfrak{sl}_N$-modules.

An essential property of the elements C_V is that their action on the Verma module $M(\lambda)$ with highest weight λ is given by the constant [3] (see also [13, Proposition 3.1.22] for the special case $q \in (0,1)$)

$$C_V|_{M(\lambda)} = \sum_{\mu \in P} (\dim V_\mu) q^{-2(\mu, \lambda+\rho)},$$

where P is the weight lattice of $U_q\mathfrak{sl}_N$, V_μ is the subspace of μ-weight vectors in V, and ρ is the half-sum of positive roots of $U_q\mathfrak{sl}_N$. Hence the same formula is applicable to any highest weight $U_q\mathfrak{sl}_N$-module with highest weight λ.

A routine verification that involves (3.3), (3.4), and (6.5) shows that for $s = k \in \mathbb{Z}_+$ the vectors $t_1^k t_N^{*k} \in \mathcal{E}_{2s}(\Xi_{n,m})_q$ are also $U_q\mathfrak{sl}_N$-singular (annihilated by E_n), and thus generate simple $U_q\mathfrak{sl}_N$-submodules with highest weights $k(\omega_1 + \omega_{N-1})$ for all $k \in \mathbb{Z}_+$.

A direct computation of those constants provides the result as follows. Let e_p be the elementary symmetric degree p polynomial in N variables. Then

$$C_{L(\omega_p)}|_{M(k(\omega_1+\omega_{N-1}))} = e_p \left(q^{-2k-N+1}, q^{-N+3}, q^{-N+5}, \ldots, q^{N-5}, q^{N-3}, q^{2k+N-1} \right).$$

On the other hand, it is clearly visible from the definitions that the matrix elements of $U_q\mathfrak{sl}_N$-actions in $\mathcal{E}_{2k}(\Xi_{n,m})_q$ with respect to a suitable PBW-basis are Laurent polynomials of q^{2k}.

Thus an analytic continuation argument implies that the collection of constants

$$e_p \left(q^{-2s-N+1}, q^{-N+3}, q^{-N+5}, \ldots, q^{N-5}, q^{N-3}, q^{2s+N-1} \right), \qquad p = 1, \ldots, N,$$

realizes as action of the central elements $C_{L(\omega_p)}$ on some non-zero simple submodules of the $U_q\mathfrak{sl}_N$-modules $\mathcal{E}_{2s}(\Xi_{n,m})_q$.

Hence for isomorphic $\mathcal{E}_{2s}(\Xi_{n,m})_q$ and $\mathcal{E}_{2s'}(\Xi_{n,m})_q$, one should have the latter collection of constants identical. This already implies that the collection of constants

$$q^{-2s-N+1}, q^{-N+3}, q^{-N+5}, \ldots, q^{N-5}, q^{N-3}, q^{2s+N-1}$$

must be also identical, which means that, given such pair s, s' then either $q^{-2s-N+1} = q^{-2s'-N+1}$ or $q^{-2s-N+1} = q^{2s'+N-1}$. We obtain

Proposition 7.6 *Given $s \in \mathbb{C}$, the set of those s' for which $\mathcal{E}_{2s'}(\Xi_{n,m})_q$ is isomorphic to $\mathcal{E}_{2s}(\Xi_{n,m})_q$ as $U_q\mathfrak{sl}_N$-modules, is a subset of*

$$\left\{ s + \frac{\pi i n}{\ln q} \middle| \; n \in \mathbb{Z} \right\} \cup \left\{ -(s+N-1) + \frac{\pi i n}{\ln q} \middle| \; n \in \mathbb{Z} \right\}.$$

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